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ADIABATIC INVARIANCE AND TRANSIENT RESONANCE IN VERY SLOWLY VARYING OSCILLATORY HAMILTONIAN SYSTEMS*

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Abstract. Canonical averaging techniques are applied to very slowly varying oscillatory systems in Hamiltonian standard form to very high orders, which are required for uniformly valid solutions. When resonance is exhibited in these systems $N-1$ adiabatic invariants are found, reducing the original system of $2N$ first-order differential equations to two differential equations that embody the resonance behavior. Three examples exhibiting transient resonance are examined.

Transient resonance occurs when the leading-order frequency of the reduced system makes a slow passage through zero. Depending on the rate of this slow passage, three distinguished cases are identified: the subcritical, the critical, and the supercritical. For each case, asymptotic solutions are found illustrating the nature of the resonance for certain classes of problems. In both the critical and supercritical cases, the action (and correspondingly the energy) can undergo changes of $O(1)$ or greater. Specific examples are used to illustrate and numerically verify all results.

Key words. adiabatic invariants, averaging, Hamiltonian systems, near-identity transformations, transient resonance

AMS(MOS) subject classification. 34E15

1. Introduction. This is a continuation of the work reported in Bosley and Kevorkian [5] on very slowly varying oscillatory Hamiltonian systems; here we concentrate on the class of problems that exhibit transient resonance.

We consider the Hamiltonian system of $2N$ differential equations

$$\frac{dq_n}{dt}=\frac{\partial h}{\partial p_n}=\omega_n(p_i,t^*)+\varepsilon g_n(p_i,q_i,t^*;\varepsilon), \tag{1.1a}$$

$$\frac{dp_n}{dt}=-\frac{\partial h}{\partial q_n}=\varepsilon f_n(p_i,q_i,t^*;\varepsilon), \qquad n=n=1,2,\cdots,N, \tag{1.1b}$$

where the Hamiltonian h has the form

$$h=h_0(p_i,t^*)+\varepsilon h^*(p_i,q_i,t^*;\varepsilon). \tag{1.2}$$

The slow time is $t^*=\varepsilon^2 t$, and ε is a small parameter, $0<\varepsilon\ll 1$. Thus, $\omega_n=\partial h_0/\partial p_n$, $g_n=\partial h^*/\partial p_n, f_n=-\partial h^*/\partial q_n$, and we assume that the functions f_n and g_n are periodic in each of the q_i with the same period 2π.

Generally, any system (not necessarily Hamiltonian) of M equations in the form (1.1a), and N equations in the form (1.1b) is said to be in *standard form* if the ω_n are independent of the q_i, and the f_n and g_n are 2π-periodic functions of each of the q_i. Although we do not consider the details here, our approach (as in the case discussed in [12]) generalizes in a straightforward manner to non-Hamiltonian systems.

Periodicity with respect to the q_i implies that h^* may be expressed as a Fourier series (either finite or infinite) of the form

$$h^*=\sum_{r_1=-\infty}^{\infty}\cdots\sum_{r_N=-\infty}^{\infty} h^*_{r_1r_2\cdots r_N}(p_i,t^*;\varepsilon)\exp i(r_1q_1+r_2q_2+\cdots+r_Nq_N). \tag{1.3}$$

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Consider a typical oscillatory term in this series, say,

$$\alpha(p_i, t^*; \varepsilon) \exp i(r_1 q_1 + r_2 q_2 + \cdots + r_R q_R),$$

where now we let $r_1, r_2, \cdots, r_R$ be nonzero integers. Note, by a simple renumbering of the variables, that any particular term in the series (1.3) can be redefined using only the first $R \leqq N$ angle variables, $q_1, q_2, \cdots, q_R$. This term will lead to a resonance between the variables $q_1, q_2, \cdots, q_R$ if the associated frequency combination

$$\sigma \equiv r_1\omega_1 + r_2\omega_2 + \cdots + r_R\omega_R \tag{1.4}$$

vanishes at some critical time $t^* = t_0^*$.

As is well known (for example, see [12]) the usual approach for calculating a solution by averaging near-identity transformations breaks down for times in some neighborhood of t_0^*. If the condition $\sigma \approx 0$ persists for times $t^* - t_0^* = O(1)$ or longer, the solution is said to be in *sustained resonance*, and this is discussed in [5]. Here we focus on *transient resonance*, the case where σ makes a single slow passage through zero.

The above problem for the case where h depends on the slow time $\tilde{t} = \varepsilon t$ has been thoroughly discussed in [12] and the references cited there. The basic idea is to isolate the critical combination of angle variables

$$\bar{q}_1 \equiv q_1 + \frac{r_2}{r_1} q_2 + \cdots + \frac{r_R}{r_1} q_R, \tag{1.5}$$

corresponding to the resonance (1.4), then remove the $N-1$ remaining angle variables from the Hamiltonian to any desired order by means of a near-identity averaging transformation. As a result, the $N-1$ momenta, corresponding to the removed variables, are adiabatic invariants, and the transformed Hamiltonian depends only on the $\bar{q}_1$ angle variable. This defines a reduced problem of order two that embodies the resonance behavior.

The generalization to a dependence on the slower time $t^* = \varepsilon^2 t$ is not trivial. The principal difficulty is that expansions are required to very high orders to derive uniformly valid results for the very long times ($t = O(1/\varepsilon^2)$) considered. Also, in the case of transient resonance the reduced problem often cannot be solved analytically. This was pointed out in [10] and will be discussed fully in § 3 of this paper.

In some problems it is possible to have two or more frequency combinations of the form (1.4) that vanish simultaneously. This feature is pointed out in [12] but is not discussed in detail here. Because the reduced problem for such systems is of order four or higher, chaotic behavior may occur, as discussed by Hoveijn and Verhulst [8] for an example where the frequencies are constants. For a more general discussion of averaging methods, see Sanders and Verhulst [15]. In particular, [15, Chap. 5] contains a discussion of systems with slowly varying frequencies and passage through resonance, and [15, Chap. 7] is a discussion of various fixed resonances for constant frequency Hamiltonian systems. Additional results for transient resonance and a proof of the validity of certain asymptotic solutions can be found in Sanders [14]. A discussion of the rigorous background and mathematical theory for averaging methods can be found in Lochak and Meunier [13] and the references cited there.

Numerical integration of rapidly oscillating functions over long times is very difficult. For smaller values of ε, numerical integration not only yields inaccurate results but requires an impractical amount of time. On the other hand, due to the very large number of terms, high-order asymptotic expansions often cannot be calculated by hand. The solution technique presented here and in [5] relies on the use of symbolic manipulation programs, which not only keep track of these terms but can be programmed to automatically execute a significant portion of the solution procedure. While still containing a large number of terms, the resulting solution can nevertheless be easily evaluated with the symbolic manipulator's automatic generation of FORTRAN code directly from the derived expressions.

In § 2, we review the averaging procedure basic to this work. This procedure is discussed in [5] for a general Hamiltonian system in the absence of any resonances. Section 2.1 summarizes the modifications necessary to this procedure when a single resonance is present. As for the case where h depends on εt, the original problem defined by a system of $2N$ first-order differential equations can be reduced by an averaging procedure to a system of two first-order equations; this is accomplished with the identification of $N-1$ adiabatic invariants that are valid through the resonance. The reader is referred to [4] for more details. In § 2.2, we consider three examples to further illustrate and numerically verify the procedure. The role of symbolic manipulation is indicated throughout.

In § 3, we examine the reduced problem for transient resonance resulting from the averaging procedure and examine several cases that correspond to different behaviors for the leading-order frequency across the resonance region. Three cases are distinguished: the subcritical, the critical, and the supercritical.

In the subcritical case, an asymptotic solution valid across the resonance for a general class of problems is found. Also, we introduce an improved matching procedure that results in a more accurate postresonance solution. This procedure has a wider range of applicability than in the present context.

In both the critical and supercritical cases, the action undergoes changes of $O(1)$ across the resonance, which often creates difficulties associated with the interior layer solution. For certain problems in the critical case, an interior layer solution is available and can be matched to the solutions away from resonance. In the supercritical case, even though solutions to the interior layer equations are available, these cannot be matched to the solution away from resonance, prohibiting a valid solution in the resonance and postresonance regions. These difficulties were not encountered for the $\tilde{t}=\varepsilon t$ case.

2. Averaging for very slowly varying oscillatory Hamiltonian systems. The procedure for averaging the system of equations (1.1) is discussed in detail in Kevorkian [12, § 4] for the case of slow variations depending on $\tilde{t}=\varepsilon t$. In [5, § 2], Bosley and Kevorkian extend the results for a general oscillatory Hamiltonian system to the necessary orders for uniformly valid solutions for the case of very slow variations depending on $t^*=\varepsilon^2 t$. Here, we focus on the case where a solution passes through a given resonance, summarizing the results in [5, § 2.2] for reference purposes.

2.1. Adiabatic invariants and the reduced problem for a given resonance. As is well known (e.g., see [12]), the ususal procedure for solving the system (1.1) by averaging near-identity transformations breaks down in a neighborhood of the resonance. Resonance is defined as the vanishing of σ, the following critical combination of the $O(1)$ frequencies of (1.1):

$$\sigma(p_i, t^*)\equiv r_1\omega_1+r_2\omega_2+\cdots+r_R\omega_R, \tag{2.1}$$

where $r_1, r_2, \cdots, r_R$ are R nonzero integers. If the ω_n depend only on t^*, then (2.1) is an *a priori* known function of t^*, and its vanishing corresponds to some critical time $t^*=t_0^*$. In the general case where the ω_n depend on the p_i, (2.1) is evaluated along a solution of (1.1).

Singularities arise in the form of σ-divisors in the solution derived by the ususal averaging procedure. Therefore, when a resonance is present in (1.1) we first isolate the critical combination of variables associated with this resonance into a single angle variable before eliminating the rest of the q_i. We define a time-independent canonical transformation from the (p_i, q_i) to new variables $(\bar{p}_i, \bar{q}_i)$ given by

$$(2.2)\qquad \begin{array}{ll} \bar{q}_1 = q_1 + \dfrac{r_2}{r_1} q_2 + \cdots + \dfrac{r_R}{r_1} q_R, & \bar{p}_1 = p_1, \\ \bar{q}_2, = q_2, & \bar{p}_2 = p_2 - \dfrac{r_2}{r_1} p_1, \\ & \vdots \\ \vdots & \bar{p}_R = p_R - \dfrac{r_R}{r_1} p_1, \\ & \bar{p}_{R+1} = p_{R+1}, \\ & \vdots \\ \bar{q}_N, = q_N, & \bar{p}_N = p_N, \end{array}$$

which isolates the resonance into a single variable, $\bar{q}_1$, allowing elimination of the rest of the $\bar{q}_i$ from the Hamiltonian. The new Hamiltonian is simply the old Hamiltonian with the (p_i, q_i) expressed in terms of the $(\bar{p}_i, \bar{q}_i)$. Because this transformation has isolated the resonance variable, this new Hamiltonian can be written in the form

$$(2.3)\qquad \begin{aligned} \bar{h}(\bar{p}_i, \bar{q}_i, t^*; \varepsilon) = {} & \underline{\bar{h}}_0(\bar{p}_i, t^*) + \varepsilon[\underline{\bar{h}}_1(\bar{p}_i, t^*) + \hat{\bar{h}}_{1c}(\bar{p}_i, \bar{q}_1, t^*) + \hat{\bar{h}}_{1s}(\bar{p}_i, \bar{q}_i, t^*)] \\ & + \varepsilon^2[\underline{\bar{h}}_2(\bar{p}_i, t^*) + \hat{\bar{h}}_{2c}(\bar{p}_i, \bar{q}_1, t^*) + \hat{\bar{h}}_{2s}(\bar{p}_i, \bar{q}_i, t^*)] + O(\varepsilon^3). \end{aligned}$$

The underbar represents the averaged part of the Hamiltonian, while the underhat indicates a zero average over the $\bar{q}_i$. The critical terms, indicated by a subscript c, contain all resonant behavior associated with $\bar{q}_1$, while the remainder of the oscillatory part, indicated by a subscript s, contains all nonresonant terms.

The averaging procedure is based on finding the generating function $F(P_i, \bar{q}_i, t^*; \varepsilon)$ of a near-identity canonical transformation to new variables (P_i, Q_i) so that the transformed Hamiltonian $H(P_i, Q_i, t^*; \varepsilon)$ is independent of $Q_2, \cdots, Q_N$. The generating function is assumed in the expanded form

$$(2.4)\qquad F(P_i, \bar{q}_i, t^*; \varepsilon) = \sum_{i=1}^{N} P_i\bar{q}_i + \varepsilon F_1(P_i, \bar{q}_i, t^*) + \varepsilon^2 F_2(P_i, \bar{q}_i, t^*),$$

and F_1, F_2 are chosen to eliminate all nonresonant terms from the Hamiltonian (2.3) to $O(\varepsilon^2)$. The final Hamiltonin takes the form

$$(2.5a)\qquad \begin{aligned} H(P_i, Q_i, t^*; \varepsilon) = {} & \underline{H}_0(P_i, t^*) + \varepsilon[\underline{H}_1(P_i, t^*) + \hat{H}_{1c}(P_i, Q_1, t^*)] \\ & + \varepsilon^2[\underline{H}_2(P_i, t^*) + \hat{H}_{2c}(P_i, Q_1, t^*)] + \varepsilon^3[\underline{H}_3(P_i, t^*) \\ & + \hat{H}_3(P_i, Q_i, t^*)] + O(\varepsilon^4), \end{aligned}$$

where

$$(2.5b)\qquad \underline{H}_0(P_i, t^*) = \underline{\bar{h}}_0(P_i, t^*),$$

$$(2.5c)\qquad \underline{H}_1(P_i, t^*) = \underline{\bar{h}}_1(P_i, t^*),$$

$$(2.5d)\qquad \hat{H}_{1c}(P_i, Q_1, t^*) = \hat{\bar{h}}_{1c}(P_i, Q_1, t^*),$$

$$(2.5e)\qquad \underline{H}_2(P_i, t^*) = \underline{\bar{h}}_2(P_i, t^*) + \underline{Z}_2(P_i, t^*),$$

$$H_{2c}(P_i, Q_1, t^*) = \bar{h}_{2c}(P_i, Q_1, t^*) + Z_{2c}(P_i, Q_1, t^*) - \frac{\partial \bar{h}_{1c}}{\partial \bar{q}_1}\frac{\partial F_1}{\partial P_1}, \tag{2.5f}$$

and

$$H_3(P_i, t^*) = \bar{h}_3(P_i, t^*) + Z_3(P_i, t^*) + \frac{\partial F_1}{\partial t^*} \equiv 0. \tag{2.5g}$$

The functions Z_2 and Z_3 are defined in [5] and are not repeated here. The details of these calculations, including the expressions for F_1 and F_2, are given for three specific examples later. The general formulas are found in [4].

Note that H_{2c} given in (2.5f) contains reference to an arbitrary averaged function F_1. The choice of F_1 therefore is significant in two ways: in accounting for averaged terms of $O(\varepsilon^3)$, which become $O(\varepsilon)$ over long times (t^* of $O(1)$), and as a contribution to the resonant terms in the $O(\varepsilon^2)$ Hamiltonian.

Because $Q_2, Q_3, \cdots, Q_N$ are removed from the Hamiltonian to $O(\varepsilon^2)$, the associated conjugate momenta $P_2, P_3, \cdots, P_N$ are constants to $O(\varepsilon^2)$. In the case of resonance, these are the $N-1$ adiabatic invariants *that remain valid through the resonance*; they are constant to $O(\varepsilon^2)$ for times $t \leqq T$, with $T = O(1/\varepsilon^2)$. By inverting the near-identity and isolating transformations, we find the adiabatic invariants as functions of the original variables (p_i, q_i),

$$\begin{aligned} P_n \equiv \mathcal{A}_n(p_i, q_i, t^*; \varepsilon) &= p_n - \frac{r_n}{r_1} p_1 - \varepsilon \frac{\partial F_1}{\partial \bar{q}_n}(\bar{p}_i, \bar{q}_i, t^*) \\ &\quad + \varepsilon^2 \left(-\frac{\partial F_2}{\partial \bar{q}_n}(\bar{p}_i, \bar{q}_i, t^*) + \sum_{j=1}^{N} \frac{\partial^2 F_1}{\partial \bar{q}_n \partial P_j}\frac{\partial F_1}{\partial \bar{q}_j}\right) \\ &= \text{constant} + O(\varepsilon^3) \end{aligned} \tag{2.6}$$

for $n = 2, 3, \cdots, N$, where the evaluations of the partial derivatives at $(\bar{p}_i, \bar{q}_i)$ are replaced by the substitutions (2.2) (note that $r_n = 0$ for $n > R$). This function of the original variables is a constant of the motion to $O(\varepsilon^2)$ along solution trajectories. In § 2.2, we illustrate this feature for two different examples.

One notable advantage of this solution technique is its suitability for symbolic manipulation programs. The entire procedure just outlined in § 2.1 can be fully automated on SMP. After entering the Hamiltonian in standard form, the programmed steps include isolating the resonance, eliminating all nonresonant terms by solving for the $O(\varepsilon)$ and $O(\varepsilon^2)$ generating functions, solving for the averaged $O(\varepsilon^2)$ terms to determine F_1, simplifying the final Hamiltonian, and inverting all the transformations for an asymptotic solution of the adiabatic invariants in terms of the original variables. On a VAXserver 3500 running Ultrix-32 v3.0 with 12 MB of RAM and 51 MB of virtual memory, SMP version 1.5 took approximately 20 minutes of CPU time to execute this procedure for the problem described in § 2.2.1. The peak amount of memory required was approximately 12 MB. This procedure probably would have taken less time and have required less memory if run on later versions of SMP or perhaps on another symbolic manipulator such as MACSYMA or Mathematica; the factorization algorithms (which are time and memory intensive operations) have been recently improved.

In contrast, the CPU times for numerical solutions of the systems of equations given in (2.2) vary according to the value of ε. For $\varepsilon = 0.01$, a numerical integration takes on the order of one to two minutes to integrate to a value for t^* of $O(1)$; for $\varepsilon = 0.001$, the time required increases by a factor of 10^4, i.e., approximately one to two weeks.

When a resonance is present we find $N-1$ adiabatic invariants (2.6), and the system of differential equations (1.1) of order $2N$ is reduced to a system of order two, for the variables P_1 and Q_1,

$$
\begin{aligned}
\frac{dQ_1}{dt} &= \sigma(P_i, t^*) + \varepsilon\left[\frac{\partial \underline{H}_1}{\partial P_1}(P_i, t^*) + \frac{\partial \underset{\wedge}{H}_{1c}}{\partial P_1}(P_i, Q_1, t^*)\right] \\
&\quad + \varepsilon^2\left[\frac{\partial \underline{H}_2}{\partial P_1}(P_i, t^*) + \frac{\partial \underset{\wedge}{H}_{2c}}{\partial P_1}(P_i, Q_1, t^*)\right] + O(\varepsilon^3),
\end{aligned} \tag{2.7a}
$$

$$
\frac{dP_1}{dt} = -\varepsilon\frac{\partial \underset{\wedge}{H}_{1c}}{\partial Q_1}(P_i, Q_1, t^*) - \varepsilon^2\frac{\partial \underset{\wedge}{H}_{2c}}{\partial Q_1}(P_i, Q_1, t^*) + O(\varepsilon^3). \tag{2.7b}
$$

This is the *reduced problem* we consider in § 3 for the case of transient resonance. The same reduced problem is discussed in [4] and [5] for the case of sustained resonance. All remaining P_n are constants and the remaining Q_n, $n = 2, 3, \cdots, N$ are found as quadratures once the solution for the system (2.7) is known. All that remains is to find uniformly valid solutions to the two coupled differential equations (2.7) throughout the resonance when $\sigma(P_i, t^*) \approx 0$.

2.2. Numerical verification of adiabatic invariance. To illustrate the procedure discussed in § 2.1 for a general Hamiltonian, we examine three examples. The analytical results are summarized for each example; for the first two examples the original action-angle systems are numerically integrated and the results substituted into (2.6) to assess the accuracy of the derived adiabatic invariants.

2.2.1. Two quadratically coupled oscillators. We consider the two weakly coupled oscillators of the form

$$
\begin{aligned}
\frac{d^2x_1}{dt^2} + \omega_1^2(t^*)x_1 &= \varepsilon x_2^2, \\
\frac{d^2x_2}{dt^2} + \omega_2^2(t^*)x_2 &= \varepsilon 2x_1x_2.
\end{aligned} \tag{2.8}
$$

These equations were studied in [10] for the case where the ω_i depend on $\tilde{t} = \varepsilon t$, and correspond to the standard form Hamiltonian

$$
\begin{aligned}
h(p_i, q_i, t^*; \varepsilon) &= \omega_1(t^*)p_1 + \omega_2(t^*)p_2 \\
&\quad + \varepsilon\frac{p_2}{2\omega_2}\sqrt{\frac{2p_1}{\omega_1}}[\sin(q_1 - 2q_2) + \sin(q_1 + 2q_2) - 2\sin(q_1)] \\
&\quad + \varepsilon^2\left[\frac{p_1}{2\omega_1}\frac{d\omega_1}{dt^*}\sin(2q_1) + \frac{p_2}{2\omega_2}\frac{d\omega_2}{dt^*}\sin(2q_2)\right].
\end{aligned} \tag{2.9}
$$

The angle variables, q_j, and the corresponding actions, p_j, are given by

$$
q_j = \arctan\left(\frac{\omega_j x_j}{\dot{x}_j}\right), \tag{2.10a}
$$

$$
p_j = \frac{\dot{x}_j^2 + \omega_j^2 x_j^2}{2\omega_j}, \tag{2.10b}
$$

and we see from (2.9) that the resonance $\sigma(t^*) = \omega_1(t^*) - 2\omega_2(t^*) = 0$ is present in the Hamiltonian to $O(\varepsilon)$. We therefore isolate the resonance combination $q_1 - 2q_2$ into a single variable $\bar{q}_1$ using the time-independent transformation

$$\begin{aligned} \bar{q}_1 &= q_1 - 2q_2, \qquad & \bar{p}_1 &= p_1, \\ \bar{q}_2 &= q_2, & \bar{p}_2 &= p_2 + 2p_1. \end{aligned} \tag{2.11}$$

This is an exact canonical transformation, which gives us the new Hamiltonian

$$\begin{aligned} \bar{h}(\bar{p}_i, \bar{q}_i, t^*; \varepsilon) = {} & [\omega_1(t^*) - 2\omega_2(t^*)]\bar{p}_1 + \omega_2(t^*)\bar{p}_2 + \varepsilon \frac{\bar{p}_2 - 2\bar{p}_1}{2\omega_2}\sqrt{\frac{2\bar{p}_1}{\omega_1}} \sin(\bar{q}_1) \\ & + \varepsilon \frac{\bar{p}_2 - 2\bar{p}_1}{2\omega_2}\sqrt{\frac{2\bar{p}_1}{\omega_1}} [\sin(\bar{q}_1 + 4\bar{q}_2) - 2\sin(\bar{q}_1 + 2\bar{q}_2)] \\ & + \varepsilon^2 \left[\frac{\bar{p}_1}{2\omega_1}\frac{d\omega_1}{dt^*}\sin(2\bar{q}_1 + 4\bar{q}_2) + \frac{\bar{p}_2 - 2\bar{p}_1}{2\omega_2}\frac{d\omega_2}{dt^*}\sin(2\bar{q}_2) \right]. \end{aligned} \tag{2.12}$$

Note that the first term in the $O(\varepsilon)$ Hamiltonian is the resonant term, which will not be eliminated by the near-identity averaging transformation. Solving for the generating function (2.4) for the averaging transformation, we find

$$\begin{aligned} F_1(P_i, \bar{q}_i, t^*) = {} & \frac{P_2 - 2P_1}{2\omega_2}\sqrt{\frac{2P_1}{\omega_1}} \\ & \cdot \left[\frac{\cos(\bar{q}_1 + 4\bar{q}_2)}{\omega_1 + 2\omega_2} - \frac{2\cos(\bar{q}_1 + 2\bar{q}_2)}{\omega_1} \right] + \underline{F}_1(P_i, t^*), \end{aligned} \tag{2.13}$$

where $\underline{F}_1$ is the undetermined function that will be chosen to eliminate all averaged terms from the $O(\varepsilon^3)$ Hamiltonian, and

$$\begin{aligned} F_2(P_i, \bar{q}_i, t^*) = {} & \frac{(4P_1^2 - P_2^2)}{16\omega_1\omega_2^2(\omega_1 + 2\omega_2)^2}\sin(2\bar{q}_1 + 8\bar{q}_2) + \frac{(P_2 - 2P_1)^2}{16\omega_1\omega_2^3(\omega_1 + 2\omega_2)}\sin(4\bar{q}_2) \\ & + \frac{(P_2 - 2P_1)P_1}{2\omega_1^2\omega_2^2(\omega_1 - \omega_2)}\sin(2\bar{q}_1 + 2\bar{q}_2) \\ & + \frac{(2P_1 - P_2)(2P_1\omega_2 - P_2(\omega_1 - \omega_2))}{4\omega_1^2\omega_2^2(\omega_1 + \omega_2)(\omega_1 + 2\omega_2)}\sin(2\bar{q}_1 + 6\bar{q}_2) \\ & - \frac{(2P_1 - P_2)(4P_1\omega_2 - P_2(\omega_1 + 2\omega_2))}{4\omega_1^3\omega_2^2(\omega_1 + 2\omega_2)}\sin(2\bar{q}_1 + 4\bar{q}_2) \\ & - \frac{(2P_1 - P_2)(P_1(4\omega_1 + 10\omega_2) - P_2(2\omega_1 + 3\omega_2))}{4\omega_1^2\omega_2^3(\omega_1 + 2\omega_2)}\sin(2\bar{q}_2) \\ & + \frac{P_1}{4\omega_1^2}\frac{d\omega_1}{dt^*}\cos(2\bar{q}_1 + 4\bar{q}_2) + \frac{P_2 - 2P_1}{4\omega_2^2}\frac{d\omega_2}{dt^*}\cos(2\bar{q}_2). \end{aligned} \tag{2.14}$$

For this problem, the $O(\varepsilon^3)$-averaged Hamiltonian is zero; therefore we set $\underline{F}_1$ equal to zero, so that only oscillatory terms remain in the $O(\varepsilon^3)$ Hamiltonian. The final Hamiltonian, which has the Q_2 dependence removed to $O(\varepsilon^2)$, is given by

$$\begin{aligned} H(P_i, Q_i, t^*; \varepsilon) = {} & \sigma(t^*)P_1 + \omega_2(t^*)P_2 + \varepsilon\frac{(P_2 - 2P_1)}{2\omega_2}\sqrt{\frac{2P_1}{\omega_1}}\sin Q_1 \\ & + \varepsilon^2 \frac{(P_2 - 2P_1)(P_1(6\omega_1 + 16\omega_2) - P_2(5\omega_1 + 8\omega_2))}{8\omega_1^2\omega_2^2(\omega_1 + 2\omega_2)} + O(\varepsilon^3), \end{aligned} \tag{2.15}$$

where $\sigma(t^*) \equiv \omega_1(t^*) - 2\omega_2(t^*)$. The elimination of Q_2 from the Hamiltonian to $O(\varepsilon^2)$, indicates that P_2 is an adiabatic invariant to $O(\varepsilon^2)$. The adiabatic invariant (2.6) in terms of the original action-angle variables becomes

$$\begin{aligned}
P_2 &= \mathcal{A}(p_i, q_i, t^*; \varepsilon) \\
&= 2p_1 + p_2 + \varepsilon \frac{2p_2}{\omega_2}\sqrt{\frac{2p_1}{\omega_1}}\left[\frac{\sin(q_1+2q_2)}{\omega_1+2\omega_2} - \frac{\sin(q_1)}{\omega_1}\right] \\
&\quad + \varepsilon^2\left[\frac{p_2(4p_1\omega_1^2 + p_2(3\omega_1^2 + 8\omega_1\omega_2 + 8\omega_2^2))}{2\omega_1^3\omega_2^2(\omega_1+2\omega_2)^2}\right. \\
&\quad + \frac{2p_1p_2}{\omega_1^2\omega_2^2(\omega_1+2\omega_2)}\cos(2q_1) - \frac{p_2^2}{4\omega_1\omega_2^3(\omega_1+2\omega_2)}\cos(4q_2) \\
&\quad - \frac{p_1p_2}{\omega_1^2\omega_2^2(\omega_1-\omega_2)}\cos(2q_1-2q_2) - \frac{p_2(4p_1\omega_2 - p_2\omega_1)}{\omega_1^2\omega_2^3(\omega_1+2\omega_2)}\cos(2q_2) \\
&\quad - \frac{p_1p_2(\omega-2\omega_2)}{\omega_1^2\omega_2^2(\omega_1+\omega_2)(\omega_1+2\omega_2)}\cos(2q_1+2q_2) \\
&\quad \left. + \frac{p_1}{\omega_1^2}\frac{d\omega_1}{dt^*}\sin(2q_1) + \frac{p_2}{2\omega_2^2}\frac{d\omega_2}{dt^*}\sin(2q_2)\right] + O(\varepsilon^3).
\end{aligned} \tag{2.16}$$

Expressions (2.11) through (2.16) have all been generated by an automated procedure programmed for use on SMP.

The function of the original variables, which is the adiabatic invariant (2.16), should remain constant to $O(\varepsilon^2)$ through the resonance. To verify this, a numerical integration of the four first-order differential equations arising from the original Hamiltonian (2.9) in the original variables (p_i, q_i, t^*) was done using differential equation subroutine DDRIV. The functions ω_1 and ω_2 were chosen to be

$$\begin{aligned}
\omega_1 &= 2 - \tfrac{1}{2}t^*, \\
\omega_2 &= \tfrac{1}{4} + \tfrac{1}{2}t^*,
\end{aligned} \tag{2.17}$$

so that $\omega_1 - 2\omega_2 = 0$ at $t^* = 1$. For Figs. 2.1 and 2.2, the other parameters include $\varepsilon = 0.05$,

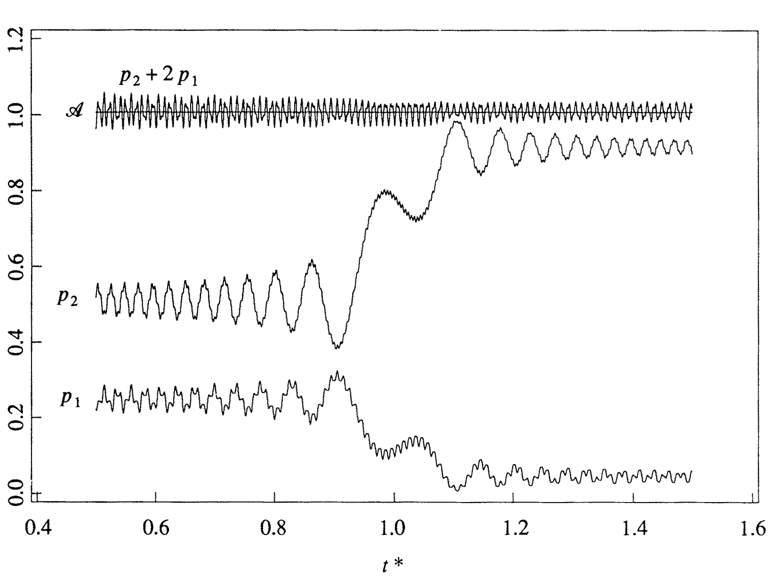

FIG. 2.1. *Passage through resonance for quadratically coupled oscillators.*

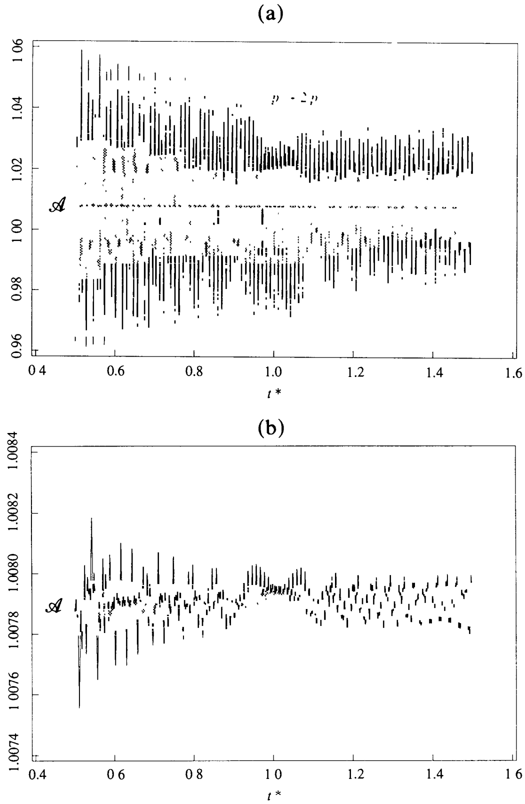

FIG. 2.2. *Adiabatic invariant for quadratically coupled oscillators.*

initial conditions $p_1 = 0.25$, $p_2 = 0.5$, $q_1 = 0.0$, $q_2 = 0.0$, and integration range $t^* = [0.0, 2.0]$. The numerical results of this integration were then substituted into the algebraic formula (2.16) for the adiabatic invariant $\mathscr{A}$, and this function plotted with the graphs for p_1, p_2, and the $O(1)$ adiabatic invariant, $2p_1 + p_2$ (see Fig. 2.1).

The function $\mathscr{A}(p_i, q_i, t^*)$ proves remarkably constant through the resonance (see Fig. 2.2(a), an enlargement of the top curve of Fig. 2.1) and in general for long times. Other values of ε and other initial conditions have been used and the same qualitative agreement has been found.

In Fig. 2.2(b) (an enlargement of the $\mathscr{A}$ curve in Fig. 2.2(a)), we see that the adiabatic invariant has an oscillatory behavior of very small magnitude due to the fact that oscillatory terms were not removed by the near-identity transformation from the $O(\varepsilon^3)$ Hamiltonian. Using several different values for ε, we can find a rough approximation for the order of the error in the adiabatic invariant (which should be $O(\varepsilon^3)$) to verify this as the cause. Assuming that the small oscillations in $\mathscr{A}$ are due to an error of $O(\varepsilon^K)$ and that $|\max \mathscr{A} - \min \mathscr{A}|$ is a linear measure of the error, we use the equation

$$|\max \mathscr{A} - \min \mathscr{A}| = C\varepsilon^K \tag{2.18}$$

to find the order K. We integrate the differential equations numerically over $t = [0, 40]$ for $\varepsilon = .05$, .02, .01, .005, and .002 and use the computed values of $\mathscr{A}$ and the known values of ε to solve for C and K in a least squares log-log fit (see Fig. 2.3).

Using the same initial conditions as for Fig. 2.1, we find that $K = 3.0023$ (with $C = 29.30$), in excellent agreement with the theoretical result of $O(\varepsilon^3)$. The value of C is large due to the great number of terms in the $O(\varepsilon^3)$ Hamiltonian, even for this

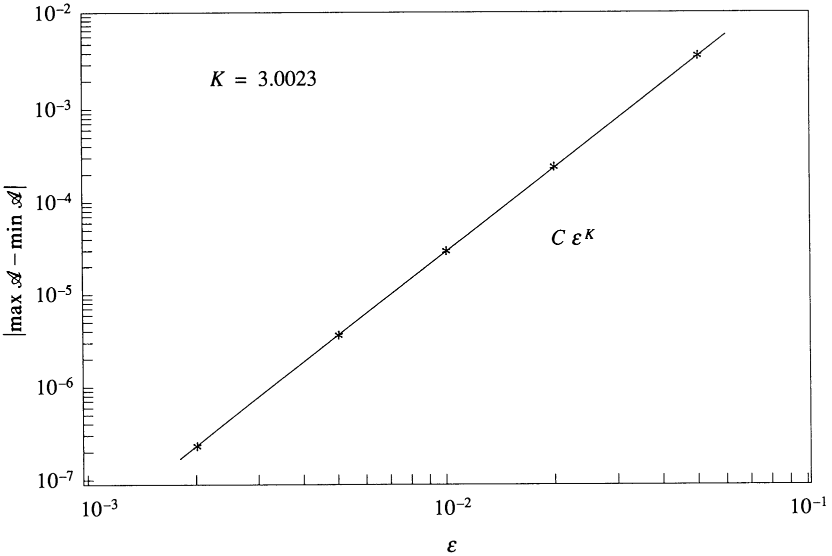

FIG. 2.3. *Order of the error in the adiabatic invariant.*

relatively simple problem. The substantial agreement between the numerical results and the analytical asymptotic result give high confidence in this method for reducing the set of differential equations.

From the final Hamiltonian, we obtain the reduced system of two coupled differential equations

$$\frac{dQ_1}{dt}=\frac{\partial H}{\partial P_1}=\sigma(t^*)+\varepsilon\frac{(\mathscr{A}-6P_1)}{2\omega_2}\sqrt{\frac{1}{2\omega_1 P_1}}\sin Q_1 \tag{2.19a}$$
$$+\varepsilon^2\left[\frac{2\mathscr{A}}{\omega_1^2\omega_2^2}-\frac{3\omega_1+8\omega_2}{\omega_1^2\omega_2^2(\omega_1+2\omega_2)}P_1\right]+O(\varepsilon^3),$$

$$\frac{dP_1}{dt}=-\frac{\partial H}{\partial Q_1}=-\varepsilon\frac{(\mathscr{A}-2P_1)}{2\omega_2}\sqrt{\frac{2P_1}{\omega_1}}\cos Q_1+O(\varepsilon^3), \tag{2.19b}$$

where P_2 has been replaced by the constant adiabatic invariant $\mathscr{A}$. This final system is equivalent to the system of three equations with one constant found in Ablowitz, Funk, and Newell [2, § 3].

2.2.2. Triad resonance for three oscillators. The present approach is the same for any number of resonant combinations. To illustrate this, we study a second example of a system of three coupled oscillators of the form

$$\frac{d^2y_i}{dt^2}+\omega_i^2(t^*)y_i=\varepsilon\kappa_i y_j y_k, \tag{2.20}$$

where $t^*=\varepsilon^2 t$; $i, j, k=1,2,3$; and the κ_i are positive $O(1)$ constants. For this system all three frequencies contribute to the possible resonances, given by

$$\omega_i+\omega_j-\omega_k=0. \tag{2.21}$$

This system has also been discussed in [2] and [7]. There has also been significant study of the resonance structure of the system of three oscillators with a more general quadratic coupling for constant frequency systems (e.g., see [15], [1], and [8]).

By noting the simple transformation, $x_i = (\kappa_j \kappa_k)^{1/2} y_i$, we set the constants $\kappa_i = 1$ without loss of generality; also, due to the symmetry in the problem, we will consider only the resonance $\omega_1 + \omega_2 - \omega_3 = 0$, as the oscillators can simply be renumbered. The system (2.20) is changed to standard form using the same transformation (2.10) as for the two coupled oscillators. The resulting Hamiltonian is

$$
\begin{aligned}
h(p_i, q_i, t^*; \varepsilon) &= \omega_1(t^*) p_1 + \omega_2(t^*) p_2 + \omega_3(t^*) p_3 \\
&\quad + \varepsilon \sqrt{\frac{p_1 p_2 p_3}{2\omega_1 \omega_2 \omega_3}} [\sin(q_1 + q_2 + q_3) \\
&\quad - \sin(q_1 + q_2 - q_3) \\
&\quad - \sin(q_1 - q_2 + q_3) + \sin(q_1 - q_2 - q_3)] \\
&\quad + \varepsilon^2 \sum_{i=1}^{3} \frac{p_i}{2\omega_i} \frac{d\omega_i}{dt^*} \sin(2q_i).
\end{aligned}
\tag{2.22}
$$

We summarize the results. To isolate the resonance, we define

$$
\begin{aligned}
\bar{q}_1 &= q_1 + q_2 - q_3, & \bar{p}_1 &= p_1, \\
\bar{q}_2, &= q_2, & \bar{p}_2 &= p_2 - p_1, \\
\bar{q}_3, &= q_3, & \bar{p}_3 &= p_3 + p_1.
\end{aligned}
\tag{2.23}
$$

Then we use the near-identity averaging procedure of § 2.1 to find the final reduced problem for Q_1 and P_1 corresponding to (2.7)

$$
\begin{aligned}
\frac{dQ_1}{dt} = \frac{\partial H}{\partial P_1} &= \sigma(t^*) + \varepsilon \frac{(3P_1^2 + 2(\mathscr{A}_2 - \mathscr{A}_3)P_1 - \mathscr{A}_2\mathscr{A}_3)}{2\sqrt{2\omega_1\omega_2\omega_3 P_1(\mathscr{A}_2 + P_1)(\mathscr{A}_3 - P_1)}} \sin Q_1 \\
&\quad + \varepsilon^2 \frac{1}{4\omega_1\omega_2\omega_3} \left[\frac{\mathscr{A}_2}{\omega_1 - \omega_2 + \omega_3} - \frac{\mathscr{A}_3}{\omega_1 + \omega_2 + \omega_3} \right. \\
&\quad \left. + P_1 \frac{(\omega_1 + \omega_2 + \omega_3)^2 - 4\omega_1\omega_2}{(\omega_1 - \omega_2 - \omega_3)(\omega_1 - \omega_2 + \omega_3)(\omega_1 + \omega_2 + \omega_3)} \right] + O(\varepsilon^3),
\end{aligned}
\tag{2.24a}
$$

$$
\frac{dP_1}{dt} = -\frac{\partial H}{\partial Q_1} = \varepsilon \sqrt{\frac{P_1(\mathscr{A}_2 + P_1)(\mathscr{A}_3 - P_1)}{2\omega_1\omega_2\omega_3}} \cos Q_1 + O(\varepsilon^3),
\tag{2.24b}
$$

where the two adiabatic invariants corresponding to P_2 and P_3 have been used in (2.24) and are given in terms of the original action-angle variables by the expressions

$$
\begin{aligned}
P_2 = \mathscr{A}_2(p_i, q_i, t^*; \varepsilon) &= p_2 - p_1 \\
&\quad - \varepsilon \sqrt{\frac{2p_1p_2p_3}{\omega_1\omega_2\omega_3}} \left[\frac{\sin(q_1 - q_2 - q_3)}{\omega_1 - \omega_2 - \omega_3} - \frac{\sin(q_1 - q_2 + q_3)}{\omega_1 - \omega_2 + \omega_3} \right] \\
&\quad + \varepsilon^2 \left[\frac{1}{4\omega_1\omega_2\omega_3} \left\{ \frac{p_1p_2 + p_1p_3 - p_2p_3}{(\omega_1 - \omega_2 - \omega_3)^2} - \frac{p_1p_2 - p_1p_3 + p_2p_3}{(\omega_1 - \omega_2 + \omega_3)^2} \right\} \right. \\
&\quad + \frac{1}{2\omega_1\omega_2\omega_3(\omega_1 - \omega_2 - \omega_3)(\omega_1 - \omega_2 + \omega_3)}
\end{aligned}
$$

$$\cdot\left\{p_3(p_2-p_1)\cos(2q_3)+\frac{p_1(p_2\omega_3+p_3\omega_1-p_3\omega_2)}{\omega_1}\cos(2q_1)\right.$$
$$\left.-\frac{p_2(p_1\omega_3-p_3\omega_1+p_3\omega_2)}{\omega_2}\cos(2q_2)\right\}$$

(2.25)

$$-\frac{1}{4\omega_1\omega_2\omega_3}\left\{\frac{p_1p_3}{(\omega_1-\omega_3)(\omega_1-\omega_2-\omega_3)}\cos[2(q_1-q_3)]\right.$$
$$+\frac{p_1p_3}{(\omega_1+\omega_3)(\omega_1-\omega_2+\omega_3)}\cos[2(q_1+q_3)]$$
$$+\frac{p_2p_3}{(\omega_2-\omega_3)(\omega_1-\omega_2+\omega_3)}\cos[2(q_2-q_3)]$$
$$+\frac{p_2p_3}{(\omega_2+\omega_3)(\omega_1-\omega_2-\omega_3)}\cos[2(q_2+q_3)]$$
$$\left.+\frac{2\omega_3p_1p_2}{(\omega_1-\omega_2)(\omega_1-\omega_2-\omega_3)(\omega_1-\omega_2+\omega_3)}\cos[2(q_1-q_2)]\right\}$$
$$\left.-\frac{p_1}{2\omega_1^2}\frac{d\omega_1}{dt^*}\sin(2q_1)+\frac{p_2}{2\omega_2^2}\frac{d\omega_2}{dt^*}\sin(2q_2)\right]+O(\varepsilon^3)$$

and

$$P_3=\mathscr{A}_3(p_i,q_i,t^*;\varepsilon)$$
$$=p_3+p_1-\varepsilon\sqrt{\frac{2p_1p_2p_3}{\omega_1\omega_2\omega_3}}\left[\frac{\sin(q_1-q_2+q_3)}{\omega_1-\omega_2+\omega_3}-\frac{\sin(q_1+q_2+q_3)}{\omega_1+\omega_2+\omega_3}\right]$$
$$+\varepsilon^2\left[\frac{1}{4\omega_1\omega_2\omega_3}\left\{\frac{p_1p_2-p_1p_3+p_2p_3}{(\omega_1-\omega_2+\omega_3)^2}+\frac{p_1p_2+p_1p_3+p_2p_3}{(\omega_1+\omega_2+\omega_3)^2}\right\}\right.$$
$$-\frac{1}{2\omega_1\omega_2\omega_3(\omega_1-\omega_2+\omega_3)(\omega_1+\omega_2+\omega_3)}$$
$$\cdot\left\{p_2(p_1+p_3)\cos(2q_2)+\frac{p_3(p_1\omega_2-p_2\omega_1-p_2\omega_3)}{\omega_3}\cos(2q_3)\right.$$
$$\left.-\frac{p_1(p_2\omega_1+p_2\omega_3-p_3\omega_2)}{\omega_1}\cos(2q_1)\right\}$$

(2.26)

$$-\frac{1}{4\omega_1\omega_2\omega_3}\left\{\frac{p_1p_2}{(\omega_1-\omega_2)(\omega_1-\omega_2+\omega_3)}\cos[2(q_1-q_2)]\right.$$
$$+\frac{p_1p_2}{(\omega_1+\omega_2)(\omega_1+\omega_2+\omega_3)}\cos[2(q_1+q_2)]$$
$$-\frac{p_2p_3}{(\omega_2-\omega_3)(\omega_1-\omega_2+\omega_3)}\cos[2(q_2-q_3)]$$
$$+\frac{p_2p_3}{(\omega_2+\omega_3)(\omega_1+\omega_2+\omega_3)}\cos[2(q_2+q_3)]$$
$$\left.-\frac{2\omega_2p_1p_3}{(\omega_1+\omega_3)(\omega_1-\omega_2+\omega_3)(\omega_1+\omega_2+\omega_3)}\cos[2(q_1+q_3)]\right\}$$
$$\left.+\frac{p_1}{2\omega_1^2}\frac{d\omega_1}{dt^*}\sin(2q_1)+\frac{p_3}{2\omega_3^2}\frac{d\omega_3}{dt^*}\sin(2q_3)\right]+O(\varepsilon^3).$$

We note from the above expressions that our results may have other potential zero divisors; we have only removed the divisor $\sigma = \omega_1 + \omega_2 - \omega_3$. For given functions ω_1, ω_2, and ω_3 of t^*, these divisors may vanish at individual times, or two or more may vanish simultaneously. Although it is possible to compute adiabatic invariants that remain valid for two or more sequential or simultaneous resonances, we do not discuss this procedure here. Some comments can be found in [11]. In the more general, but constant frequency systems discussed in [15] and [8], it is the presence of simultaneous resonances that leads to nonintegrability of the systems and the presence of chaos for certain frequency ratios.

To verify the formulas representing the adiabatic invariants, we numerically integrate the six first-order equations resulting from the original Hamiltonian (2.22), then substitute the results into $\mathscr{A}_2$ (2.25) and $\mathscr{A}_3$ (2.26). Using the frequencies

$$\begin{aligned} \omega_1 &= \tfrac{1}{2} + \tfrac{1}{2}t^*, \\ \omega_2 &= 1 + \tfrac{1}{2}t^*, \\ \omega_3 &= 2 + \tfrac{1}{2}t^*, \end{aligned} \tag{2.27}$$

and the initial conditions $p_1 = 0.75$, $p_2 = 0.5$, $p_3 = 0.25$, and $q_1 = q_2 = q_3 = 0$, for the value $\varepsilon = 0.05$, we integrate the interval $t^* = [0.0, 2.0]$. In Fig. 2.4 we show the interaction between the three oscillators as the system passes through resonance at $t^* = 1$.

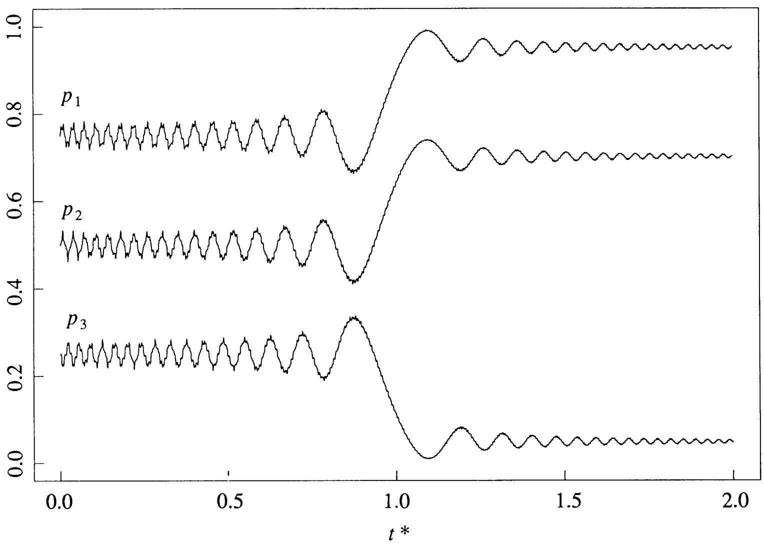

FIG. 2.4. *Passage through triad resonance.*

In Figs. 2.5 and 2.6, we illustrate the adiabatic invariants $\mathscr{A}_2$ and $\mathscr{A}_3$, on plots of $p_2 - p_1$ and $p_3 + p_1$, respectively. Although the formulas describing the adiabatic invariants have a significant number of terms, evaluation is not difficult as they were derived on the symbolic manipulator SMP from which the FORTRAN code is directly generated. Constancy of these functions to $O(\varepsilon^2)$ is again excellent as indicated.

2.2.3. Resonantly forced oscillator. Passage through resonance for the linear oscillator with slowly varying frequency and small harmonic forcing function has been discussed by Kevorkian [10] for variations depending on the slow time $\tilde{t} = \varepsilon t$. In

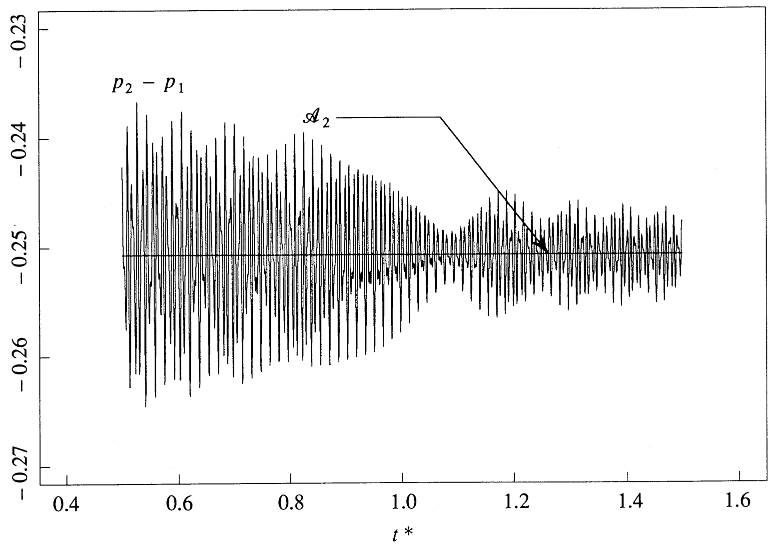

FIG. 2.5. *Adiabatic invariant* I *for triad resonance.*

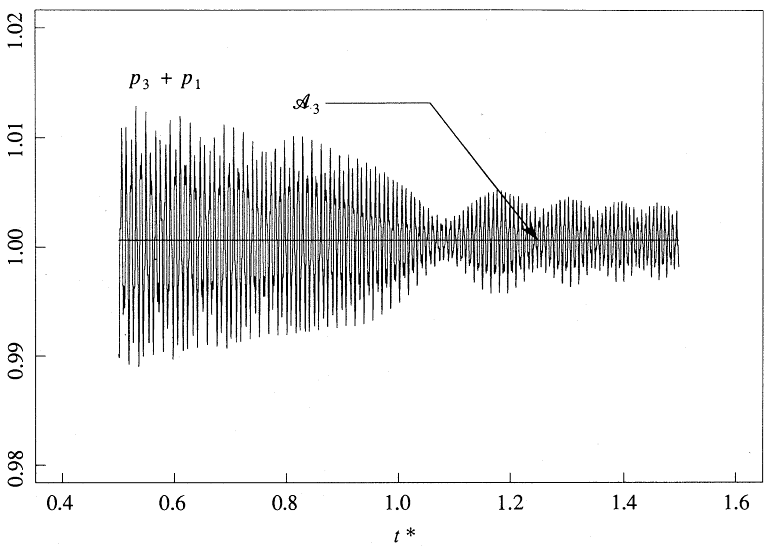

FIG. 2.6. *Adiabatic invariant* II *for triad resonance.*

addition, Kevorkian [9] discusses the solution to the resonantly forced oscillator, as does Gautesen [6] when the forcing term is $O(1)$. Here, we consider very slow passage through resonance for the oscillator

$$\ddot{x}+(1+\sigma(t^*))^2x=-\varepsilon\lambda\cos(t+\theta), \tag{2.28}$$

where we restrict $\lambda>0$ without loss of generality.

To use the averaging technique of § 2.1 and find the reduced system, we augment the equation by defining $q_2=t+\theta$, an additional angle variable, so that (2.28) follows from the Hamiltonian

$$H(x,\dot{x},q_2,p_2,t^*;\varepsilon)=\frac{\dot{x}^2+(1+\sigma)^2x^2}{2}+p_2+\varepsilon x\lambda\cos q_2. \tag{2.29}$$

We use the usual action-angle transformation for $x, \dot{x}$ given by (2.10), where p_1 and q_1 are the action and angle variables, respectively, associated with $x(t)$. The new Hamiltonian, now in standard form, is

$$
\begin{aligned}
h(q_i, p_i, t^*; \varepsilon) = {} & (1+\sigma(t^*))p_1 + p_2 \\
& + \varepsilon\lambda\sqrt{\frac{p_1}{2(1+\sigma(t^*))}}[\sin(q_1-q_2)+\sin(q_1+q_2)] \\
& + \varepsilon^2 \frac{\sigma'(t^*)p_1}{2(1+\sigma(t^*))}\sin(2q_1).
\end{aligned}
\tag{2.30}
$$

Using the procedure outlined in § 2.1, we isolate the resonance into a single variable and average the system using the near-identity transformation. The resulting reduced problem corresponding to (2.7) is then

$$\frac{dQ_1}{dt} = \sigma(t^*) + \varepsilon\frac{\lambda}{2(2P_1(1+\sigma))^{1/2}}\sin Q_1 + O(\varepsilon^3), \tag{2.31a}$$

$$\frac{dP_1}{dt} = -\varepsilon\lambda\sqrt{\frac{P_1}{2(1+\sigma)}}\cos Q_1 + O(\varepsilon^3), \tag{2.31b}$$

where the dependence of P_1 and Q_1 on the original action-angle variables is

$$Q_1 = q_1 - (t+\theta) + \varepsilon\frac{\lambda}{2(2+\sigma(t^*))}\sqrt{\frac{1}{2p_1(1+\sigma(t^*))}}\cos(q_1+t+\theta) + O(\varepsilon^2), \tag{2.32a}$$

$$P_1 = p_1 + \varepsilon\frac{\lambda}{2+\sigma(t^*)}\sqrt{\frac{p_1}{2(1+\sigma(t^*))}}\sin(q_1+t+\theta) + O(\varepsilon^2). \tag{2.32b}$$

In § 3.4, we discuss an exact solution of (2.31) if the terms of $O(\varepsilon^3)$ are ignored.

2.3. Discussion of averaging results. For a complete solution to the problems mentioned in § 2.2, we must find uniformly valid solutions (though the resonance region) to the reduced systems of equations (2.19), (2.24) and (2.31). It is important to note that although we have only discussed weakly nonlinear examples here, the results obtained from the averaging procedure of § 2.1 are more generally valid and also apply to strongly nonlinear systems and to constant frequency systems, once they have been transformed to the standard form (1.1). The resulting reduced system embodies the different resonant behaviors that are indicated by the strong or weak nonlinearity and the slowly varying or constant nature of the frequencies. We reiterate that the above results are restricted to one specific resonance. The solution of the reduced problem for the transient resonance exhibited here is addressed in the next section. For a solution to the reduced system when sustained resonance is present (not possible for weakly nonlinear systems), see Bosley and Kevorkian [5].

The solution of the resonance equations present significant difficulty since an $O(1)$ change in the action (and also in the energy) occurs across the resonance for many problems. In the simpler case, where the slow time is $\tilde{t} = \varepsilon t$, the change in the action is only $O(\sqrt{\varepsilon})$ and a matching can be accomplished (see [10] and [12]). This feature is illustrated in Fig. 2.7, where we note the comparison between a numerical integration of the same system of coupled oscillators (2.8) with $O(\varepsilon)$ quadratic coupling, having identical parameters and initial conditions, except for the change in the slow time variable from $\tilde{t} = \varepsilon t$ (Fig. 2.7(a)) to $t^* = \varepsilon^2 t$ (Fig. 2.7(b)). There is a significant qualitative difference in the behavior across the resonance due to the fact that the oscillators are closer to resonance for longer times ($t = O(1/\varepsilon)$ as compared to $t = O(1\sqrt{\varepsilon})$);

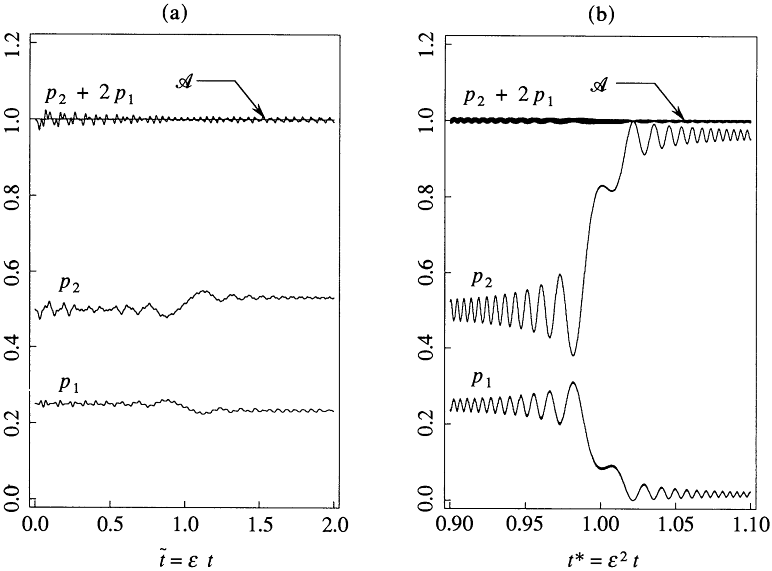

FIG. 2.7. *Effect of the slow time on resonant energy transfer.*

instead of a small energy transfer from one oscillator to the other, the transfer in energy is almost total. The figures are plotted so that the resonance layers have the same thickness.

3. The reduced problem and transient resonance. In this section we examine the reduced problem resulting from the averaging procedure of § 2. The system (2.7) of two coupled differential equations obtained is dependent on P_1, Q_1, t^*, and the $N-1$ adiabatic invariants found by eliminating the remaining Q_i. This system exhibits resonance when the $O(1)$ frequency $\sigma(P_1, t^*)$ vanishes. Two qualitatively different behaviors can result from this condition. The first, examined in this section, is that of transient resonance where σ makes a slow passage through zero vanishing at some time t_0^*. This type of resonance has several interesting features, most notably that the action (and correspondingly the energy) can undergo changes of $O(1)$ or greater across the resonance.

The second phenomenon examined in [4] and [5] is sustained resonance, where σ oscillates aout zero for long times (t^* of $O(1)$). This can occur only when the frequency is a function of the momentum P, indicating that sustained resonance is a strictly nonlinear phenomenon; therefore the weakly nonlinear oscillators examined in § 2.2 cannot exhibit sustained resonance.

Much study of resonance in Hamiltonian systems is done for the case of constant frequencies ω_i (e.g., [15], [1], and [8]). Of course, in this case, *any* resonance is *always* a sustained resonance. For constant frequencies, in the presence of a single resonance, the averaging procedure in § 2 still reduces the system of $2N$ equations to two equations embodying the resonance behavior. This reduced system does permit an asymptotic solution, completing the solution for the constant frequency case. The reason for the chaotic behavior observed by Hoveijn and Verhulst [8] is due to the presence of two (or more) simultaneous resonances. For two simultaneous resonances, only $N-2$ adiabatic invariants result from the averaging procedure. Consequently, the reduced system, which then consists of four equations, may admit chaotic behavior.

The reduced system of differential equations embodies the resonance behavior, and although canonical averaging is no longer possible, the system is still Hamiltonian.

The reduced problems resulting from the systems discussed in § 2.2 are all examples derivable from a Hamiltonian of the form

$$H(P, Q, t^*; \varepsilon) = H_0(P, t^*) + \varepsilon A(P, t^*) \sin Q + \varepsilon^2 B(P, t^*) + O(\varepsilon^3), \tag{3.1}$$

where for transient resonance we assume that H_0 is linear in P

$$H_0(P, t^*) = \sigma(t^*) P \tag{3.2}$$

so that the $O(1)$ frequency is a function of t^* alone. (Note that the subscript 1 has been dropped.) The system of equations that results is

$$\dot{Q} = \sigma(t^*) + \varepsilon \frac{\partial A}{\partial P}(P, t^*) \sin Q + \varepsilon^2 \frac{\partial B}{\partial P}(P, t^*), \tag{3.3a}$$

$$\dot{P} = -\varepsilon A(P, t^*) \cos Q, \tag{3.3b}$$

where ε is a small parameter $0 < \varepsilon \ll 1$, $t^* = \varepsilon^2 t$, and $\sigma(t^*)$ vanishes at t_0^*. System (3.3) contains the minimal features necessary to examine transient resonance for standard-form Hamiltonian systems.

Although transient resonance is possible for nonlinear problems when σ depends on P, the details are not discussed here. We merely note that there are qualitative differences from the $\tilde{t} = \varepsilon t$ case solved by Kevorkian in [12] and from the linear cases discussed in the following sections.

3.1. Solution of the reduced problem away from the resonance. We assume that σ vanishes at some time t_0^*, and therefore must consider the solution of this problem both outside the resonance region ($\sigma \neq 0$) and inside the resonance region ($\sigma \approx 0$). If $\sigma(t^*)$ is bounded away from zero, we can solve the system of two coupled ordinary differential equations (3.3) by a multiple scales procedure, a direct generalization of the procedure discussed in [12, § 6.1.1] for the case where the slow time is $\tilde{t} = \varepsilon t$. We consider the solution to be a function of two independent timescales: a fast time τ, defined by

$$\frac{d\tau}{dt} = \sigma(t^*) + \varepsilon \nu(t^*) + \varepsilon^2 \mu(t^*) + \varepsilon^3 \eta(t^*) + O(\varepsilon^4), \tag{3.4}$$

where $\nu(t^*)$, $\mu(t^*)$, and $\eta(t^*)$ are to be determined by consistency conditions, and the slow timescale $t^* = \varepsilon^2 t$. The time derivative is then

$$\frac{d}{dt} = \frac{\partial}{\partial \tau} \frac{d\tau}{dt} + \varepsilon^2 \frac{\partial}{\partial t^*}. \tag{3.5}$$

We assume expansions for Q and P of the form

$$Q = \tau + \varepsilon q^{(1)}(\tau, t^*) + \varepsilon^2 q^{(2)}(\tau, t^*) + \cdots, \tag{3.6a}$$

$$P = p^{(0)}(t^*) + \varepsilon p^{(1)}(\tau, t^*) + \varepsilon^2 p^{(2)}(\tau, t^*) + \cdots, \tag{3.6b}$$

so that the $O(1)$ differential equations are automatically satisfied. Suubstituting these expansions into the differential equations (3.3), using the time derivative (3.5), and Taylor expanding the right-hand side about τ and $p^{(0)}$, we find a sequence of differential equations to be solved for the $q^{(j)}$ and $p^{(j)}$. Since the multiple scales procedure is straightforward, we summarize the solution to (3.3) away from the resonance region.

The expansions valid to $O(\varepsilon^2)$ are

(3.7a)
$$Q(\tau, t^*) = \tau - \varepsilon \frac{1}{\sigma(t^*)} \frac{\partial A}{\partial P}(p^{(0)}, t^*) \cos \tau + \varepsilon^2 \left[\frac{1}{4\sigma^2(t^*)} \left(A \frac{\partial^2 A}{\partial P^2} - \left(\frac{\partial A}{\partial P} \right)^2 \right) \sin 2\tau + \bar{q}^{(2)}(t^*) \right] + O(\varepsilon^3),$$

(3.7b)
$$P(\tau, t^*) = p^{(0)} - \varepsilon \frac{A(p^{(0)}, t^*)}{\sigma(t^*)} \sin \tau + \varepsilon^2 \bar{p}^{(2)}(t^*) + O(\varepsilon^3),$$

where

(3.7c)
$$\tau = \tau_0 + \frac{1}{\varepsilon^2} \int_{t_c^*}^{t^*} \sigma(s)\, ds + \int_{t_c^*}^{t^*} \mu(s)\, ds + O(\varepsilon^2).$$

The lower limit of integration t_c^* is chosen to correspond to either the pre- or postresonance expansion. In the case of the preresonance expansion, the constants $p^{(0)}$ and τ_0 are found from initial conditions, whereas for the postresonance expansion these are derived from matching with the interior layer solution. The constants of integration for the higher-order terms in (3.6) are incorporated into the $O(1)$ terms, and the functions $\bar{p}^{(2)}$ and $\bar{q}^{(2)}$ are defined from the $O(\varepsilon^4)$ equations and are not given. In addition, consistency of the solution with respect to the t^* behavior gives

(3.8a)
$$\nu(t^*) = 0,$$

(3.8b)
$$\mu(t^*) = \frac{\partial B}{\partial P} - \frac{1}{2\sigma(t^*)} \left(\left(\frac{\partial A}{\partial P} \right)^2 + A \frac{\partial^2 A}{\partial P^2} \right),$$

(3.8c)
$$\eta(t^*) = 0.$$

3.2. Interior layer scaling. In the resonance region itself, where $t^* \approx t_0^*$, $\sigma(t_0^*) = 0$, we must rescale the time to get the correct $O(1)$ equations. We assume that the leading-order behavior at this zero is given by

(3.9)
$$\sigma(t^*) \sim \sigma_\alpha^\pm |t^* - t_0^*|^\alpha$$

as t^* approaches t_0^*, where $\alpha > 0$ is a given constant, and $\sigma_\alpha^\pm$ are also constants for $t^* > t_0^*$ and $t^* < t_0^*$, respectively, defined to make the passage "smooth." From this behavior, we can derive the new leading-order problem in the resonance region near $t^* = t_0^*$ by rescaling the time

(3.10)
$$t_1 = \frac{t^* - t_0^*}{\varepsilon^\delta} = \varepsilon^{2-\delta} t - \frac{t_0^*}{\varepsilon^\delta}$$

for some constant $\delta > 0$, and considering P and Q to be functions of the new time t_1. These choices transform the differential equations (3.3) to

(3.11a)
$$\frac{dQ}{dt_1} = \varepsilon^{\delta\alpha - 2 + \delta} \sigma_\alpha^\pm |t_1|^\alpha + \varepsilon^{\delta - 1} \frac{\partial A}{\partial P}(P, t^*) \sin Q + \text{higher-order terms},$$

(3.11b)
$$\frac{dP}{dt_1} = -\varepsilon^{\delta - 1} A(P, t^*) \cos Q + \text{higher-order terms}.$$

Two considerations are important in determining the new time scaling. First, we desire the richest $O(1)$ equations, leading us to choose, in accordance with (3.11), the greater of $\delta = 1$ or $\delta = 2/(1+\alpha)$. However, the leading-order behavior (3.9) also determines the interior layer thickness, $t^* - t_0^* = O(\varepsilon^{1/\alpha})$, because the asymptotic expansions (3.7) break down when $\sigma(t^*)$ becomes of $O(\varepsilon)$. Therefore, matching cannot be accomplished unless the interior layer solution extends to where the outer solution breaks down. This condition requires that

$$\delta \leqq \frac{1}{\alpha}. \tag{3.12}$$

For $0 < \alpha < 1$, called the subcritical case, the choice of $\delta = 2/(1+\alpha)$ is the greater of the two values and satisfies the matching restriction (3.12). Thus, an asymptotic solution and matching are possible (as will be shown in § 3.3).

The case where $\alpha = 1$, where the leading-order behavior is

$$\sigma(t^*) \sim \sigma_1(t^* - t_0^*) \tag{3.13}$$

is the critical case: $\delta = 1$ for both choices, and all the terms in (3.11) are of $O(1)$. The matching restriction is satisfied with equality; however, in this case the $O(1)$ equations are nonautonomous, nonlinearly coupled, and permit analytical solutions only for certain problems. These special cases illustrate the difficulty with the general problem, as both the action P and the phase constant τ_0 undergo $O(1)$ changes across the resonance layer. In § 3.5, we carry out the matching across the resonance using an interior layer solution found for the problem described in § 3.4 for the critical case.

The supercritical ($\alpha > 1$) case leads to a choice of $\delta = 1$, but this does not satisfy the restriction (3.12). Hence, while the leading-order equations found in this case are autonomous and have a solution, matching this solution with the expansions valid away from the resonance is not possible—there is no overlap matching region. In this case, the change in the action is also $O(1)$ or greater. In § 3.6, we illustrate the change

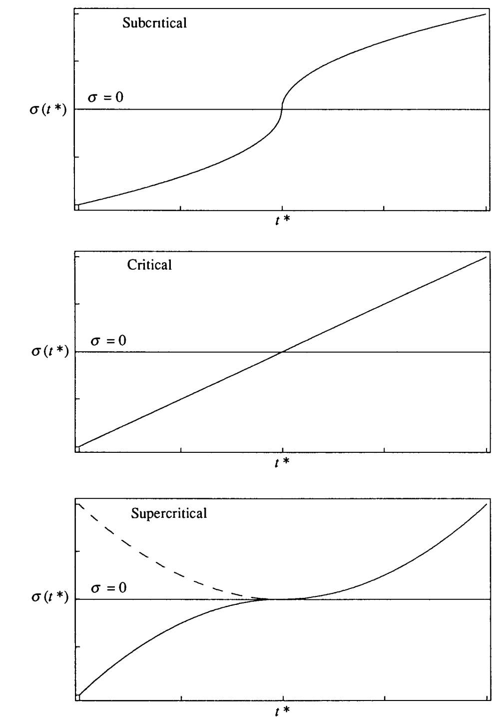

FIG. 3.1. *Qualitative behavior of $\sigma(t^*)$ across resonance.*

across the resonance for the reduced problem discussed in § 3.4, resulting from the forced oscillator (2.28) and having an exact solution.

In Fig. 3.1, we illustrate the behavior of σ across the resonance for each of the three cases. Note that for the supercritical case there are two possible qualitative behaviors with "smooth" passage through zero.

3.3. Subcritical interior layer solution. In the subcritical case, $0<\alpha<1$, we rescale the time by

$$t_1=\frac{t^*-t_0^*}{\varepsilon^{2/(1+\alpha)}}, \tag{3.14}$$

which gives the $O(1)$ equations

$$\frac{dQ}{dt_1}=\sigma_\alpha^{\pm}|t_1|^{\alpha} \tag{3.15a}$$

and

$$\frac{dP}{dt_1}=0. \tag{3.15b}$$

All higher-order terms result from a remainder of the expansion of $\sigma(t^*)$, $A(P, t^*)$, and $B(P, t^*)$ in t^* about t_0^*. To be more explicit, we choose the value $\alpha=1/2$ and find the $O(1)$ solutions and first correction terms for P and Q (note for all α, $0<\alpha<1$, the problem has the same features as this particular case). We let $\sigma(t^*)$ have the expansion

$$\sigma(t^*)=\sigma_{1/2}\operatorname{sgn}(t^*-t_0^*)|t^*-t_0^*|^{1/2}+\sigma_1(t^*-t_0^*)+\cdots. \tag{3.16}$$

For $\alpha=1/2$,

$$\frac{dt_1}{dt}=\varepsilon^{2/3} \tag{3.17}$$

so that the differential equations become

$$\frac{dQ}{dt_1}=\sigma_{1/2}\operatorname{sgn}(t_1)|t_1|^{1/2}+\varepsilon^{1/3}\frac{\partial A}{\partial P}(P, t_0^*)\sin Q+\varepsilon^{2/3}\sigma_1 t_1+O(\varepsilon^{4/3}), \tag{3.18a}$$

$$\frac{dP}{dt_1}=-\varepsilon^{1/3}A(P, t_0^*)\cos Q+O(\varepsilon^{5/3}). \tag{3.18b}$$

Letting

$$Q(t_1)=Q_0(t_1)+\varepsilon^{1/3}Q_{1/3}(t_1)+\cdots, \tag{3.19a}$$

$$P(t_1)=P_0(t_1)+\varepsilon^{1/3}P_{1/3}(t_1)+\cdots, \tag{3.19b}$$

and expanding about Q_0 and P_0 gives a sequence of differential equations that are easily integrated. As can be seen from the $O(1)$ equations (3.15), P_0 remains constant to $O(1)$ and Q_0 is a simple quadrature of the $O(1)$ frequency giving

$$Q_0(t_1)=\frac{2\sigma_{1/2}}{3}|t_1|^{3/2}+\xi_0, \tag{3.20a}$$

$$P_0(t_1)=\eta_0, \tag{3.20b}$$

where ξ_0 and η_0 are constants of the integration. Using these results, the $O(\varepsilon^{1/3})$ equations are

$$\frac{dQ_{1/3}}{dt_1}=\frac{\partial A}{\partial P}(\eta_0, t_0^*)\sin\left(\frac{2\sigma_{1/2}}{3}|t_1|^{3/2}+\xi_0\right), \tag{3.21a}$$

$$\frac{dP_{1/3}}{dt_1}=-A(\eta_0, t_0^*)\cos\left(\frac{2\sigma_{1/2}}{3}|t_1|^{3/2}+\xi_0\right), \tag{3.21b}$$

which give the following integrals as solutions:

$$Q_{1/3}(t_1)=\xi_{1/3}+\frac{\partial A}{\partial P}(\eta_0, t_0^*)\int_0^{t_1}\sin\left(\frac{2\sigma_{1/2}}{3}|\tau|^{3/2}+\xi_0\right)d\tau, \tag{3.22a}$$

$$P_{1/3}(t_1)=\eta_{1/3}-A(\eta_0, t_0^*)\int_0^{t_1}\cos\left(\frac{2\sigma_{1/2}}{3}|\tau|^{3/2}+\xi_0\right)d\tau. \tag{3.22b}$$

Note that in this case an $O(\varepsilon^{1/3})$ change can result in P across the resonance. This value is found by computing the integral found in the expressions (3.22) for the limits $-\infty$ to $+\infty$. This solution then can be matched to the expansions in the pre- and postresonance regions to determine the uniformly valid solution for all times.

We first note the value of the complex integral

$$Y=\int_{-\infty}^{+\infty}\exp\left[i\left(\frac{2\sigma_{1/2}}{3}|\tau|^{3/2}+\xi_0\right)\right]d\tau=\frac{2}{3}\left(\frac{3}{2\sigma_{1/2}}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)(1+i\sqrt{3})\,e^{i\xi_0} \tag{3.23}$$

and recognize that both its real and imaginary parts occur in (3.22). Using (3.22b) we find that

$$P_{1/3}(+\infty)=P_{1/3}(-\infty)-A(\eta_0, t_0^*)\operatorname{Re}(Y) \tag{3.24}$$

so that the change in the action across the resonance is

$$\Delta P(\eta_0, \xi_0)=-\varepsilon^{1/3}A(\eta_0, t_0^*)\frac{4}{3}\left(\frac{3}{2\sigma_{1/2}}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)\cos\left(\frac{\pi}{3}+\xi_0\right) \tag{3.25a}$$

and the change in the phase constant is

$$\Delta\tau_0(\eta_0, \xi_0)=\varepsilon^{1/3}\frac{\partial A}{\partial P}(\eta_0, t_0^*)\frac{4}{3}\left(\frac{3}{2\sigma_{1/2}}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)\sin\left(\frac{\pi}{3}+\xi_0\right). \tag{3.25b}$$

The values ξ_0 and η_0 are found by matching (3.20) and (3.22) with the outer solution for $t^*<t_0^*$ given by (3.7). Using the ususal matching procedure to $O(\varepsilon^{1/3})$, we find

$$\xi_0=\hat{\tau}_0=\tau_0^-+\frac{1}{\varepsilon^2}\int_0^{t_0^*}\sigma(t^*)\,dt^*+\int_0^{t_0^*}\mu(t^*)\,dt^*, \tag{3.26a}$$

$$\eta_0=p_-^{(0)}, \tag{3.26b}$$

$$\xi_{1/3}=\frac{\partial A}{\partial P}(p_-^{(0)}, t_0^*)\frac{2}{3}\left(\frac{3}{2\sigma_{1/2}}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)\sin\left(\frac{\pi}{3}+\hat{\tau}_0\right), \tag{3.26c}$$

$$\eta_{1/3}=-A(p_-^{(0)}, t_0^*)\frac{2}{3}\left(\frac{3}{2\sigma_{1/2}}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)\cos\left(\frac{\pi}{3}+\hat{\tau}_0\right), \tag{3.26d}$$

where τ_0^- and $p_-^{(0)}$ are constants found from initial conditions for the outer solution (3.7) in the preresonance region. Using the values of the constants in the interior layer, we can determine the values of the constants in the outer solution for $t^*>t_0^*$ by a similar matching procedure. The constants $p_+^{(0)}$ and τ_0^+ can be expressed in terms of the constants of integration for $t^* < t_0^*$ and the real and imaginary parts of the complex integral Y across the resonance region

(3.27a) $$\tau_0^+ = \hat{\tau}_0 + \varepsilon^{1/3}\frac{\partial A}{\partial P}(p_-^{(0)}, t_0^*)\frac{4}{3}\left(\frac{3}{2\sigma_{1/2}}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)\sin\left(\frac{\pi}{3}+\hat{\tau}_0\right),$$

(3.27b) $$p_+^{(0)} = p_-^{(0)} - \varepsilon^{1/3}A(p_-^{(0)}, t_0^*)\frac{4}{3}\left(\frac{3}{2\sigma_{1/2}}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)\cos\left(\frac{\pi}{3}+\hat{\tau}_0\right).$$

This completes the solution to the reduced problem valid to $O(\varepsilon^{1/3})$ throughout the domain of interest.

We can check the validity of these results by numerically integrating the original equations (3.3) using the test functions

(3.28a) $$A(P, t^*) = \frac{1}{4}(5 - t^*)\sqrt{P}(1 - 2P),$$

(3.28b) $$B(P, t^*) = \frac{P}{2}(1 + t^*)^2,$$

and

(3.28c) $$\sigma(t^*) = 3\ \text{ign}\,(t^* - 1)|t^* - 1|^{1/2}.$$

These functions are a slight modification of the two coupled oscillators discussed in § 2.2.1. Using a value of $\varepsilon = .05$ and initial conditions $Q(0) = \pi$ and $P(0) = .35$, we calculate the change in the action and phase shift across the resonance from (3.27) to find the constants for the postresonance asymptotic expansion. In Fig. 3.2 we compare the asymptotic expansions for the outer solutions in both the pre- and postresonance regions found from (3.7) with a numerical solution for P. The preresonance expansion uses constants of integration in (3.7) derived from the initial conditions given above. This expansion proves very accurate up to the resonance. The postresonance expansion in Fig. 3.2 uses the constants computed from (3.27).

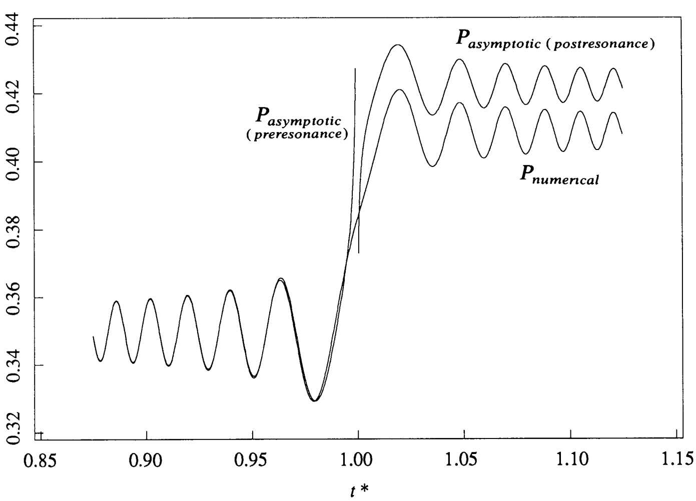

FIG. 3.2. *Subcritical asymptotic solutions using usual matching procedure.*

Note that the solution in the postresonance region is considerably less accurate than that of the preresonance solution. This error in expressions (3.27) is primarily due to the use of the constants $\hat{\tau}_0$ and $p_-^{(0)}$ in place of ξ_0 and η_0, approximating their values by their $O(1)$ behavior as $t_1 \to -\infty$. In the interior layer problem, the actual integration constants ξ_0 and η_0 are the values for Q and P at time $t_1 = 0$, so we expect an improved calculation for the postresonance constants when the functions (3.25) for ΔP and $\Delta\tau_0$ are evaluated at the true $Q(0)$ and $P(0)$. We can better approximate these true values by adding half of the expected change for the resonance layer to the $-\infty$ values

(3.29a) $$\eta_0 = p_-^{(0)} + \tfrac{1}{2}\Delta P(p_-^{(0)}, \hat{\tau}_0),$$

(3.29b) $$\xi_0 = \hat{\tau}_0 + \tfrac{1}{2}\Delta\tau_0(p_-^{(0)}, \hat{\tau}_0).$$

We then substitute (3.29) into (3.25) to get an improved "centered" calculation of $\Delta\tau_0$ and ΔP. This procedure is analogous to the modified Euler method for improving the accuracy of the simple forward difference approximation of a derivative.

These substitutions do not affect the validity of the matching, as this procedure is equivalent to specifying $\xi_{1/3} = \eta_{1/3} = 0$ and forcing the $O(1)$ constants ξ_0 and η_0 to contain the $O(\varepsilon^{1/3})$ contribution in the matching region. The results for this improved matching give

(3.30a) $$\xi_0 = \hat{\tau}_0 + \varepsilon^{1/3}\frac{\partial A}{\partial P}(p_-^{(0)}, t_0^*)\frac{2}{3}\left(\frac{3}{2\sigma_{1/2}}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)\sin\left(\frac{\pi}{3}+\hat{\tau}_0\right),$$

(3.30b) $$\eta_0 = p_-^{(0)} - \varepsilon^{1/3}A(p_-^{(0)}, t_0^*)\frac{2}{3}\left(\frac{3}{2\sigma_{1/2}}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)\cos\left(\frac{\pi}{3}+\hat{\tau}_0\right),$$

(3.30c) $$\xi_{1/3} = 0,$$

(3.30d) $$\eta_{1/3} = 0.$$

The values (3.30) are equivalent to (3.26), and both matchings are valid as $\varepsilon \to 0$; however, we expect that use of (3.30) will give quantitatively better results for the postresonance asymptotic solution. In Table 3.1 we show the values for $\Delta\tau_0$ and ΔP calculated from the usual matching procedure using the preresonance values (3.26), from the improved approximation using the centered values (3.30), and the numerical result. As expected, when we use (3.30a), (3.30b) in (3.25) substantial improvement is gained for the asymptotic solution in the postresonance region.

TABLE 3.1
Comparison of asymptotic matching with numerical solution.

	Numerical	Original matching	Error	% Error
ΔP	0.059	0.072	0.013	22%
$\Delta\tau_0$	0.137	0.092	−0.045	33%

	Numerical	Improved matching	Error	% Error
ΔP	0.059	0.057	−0.002	3%
$\Delta\tau_0$	0.137	0.134	−0.003	2%

In Fig. 3.3 we compare the asymptotic expansions for the outer solution (3.7) using the improved values, with the numerical solution for P and see the definite improvement in the accuracy of the postresonance expansion. This agreement is quite good given the large value of ε (i.e., $\varepsilon^{1/3} = (.05)^{1/3} = .368$). In Fig. 3.4, we show the difference between the numerical and the asymptotic solution for Q, using the improved values, seeing that it remains near zero except in the resonance region. To find a solution in the resonance region itself, we use the interior layer solutions (3.20) and (3.22) (not shown in Figs. 3.2 through 3.4).

For general α, $0<\alpha<1$, if we assume the leading-order behavior of $\sigma(t^*)$ as

$$\sigma(t^*) \sim \sigma_\alpha \operatorname{sgn}(t^* - t_0^*)|t^* - t_0^*|^\alpha, \tag{3.31}$$

we can easily generalize the above solutions to find the change in the action across

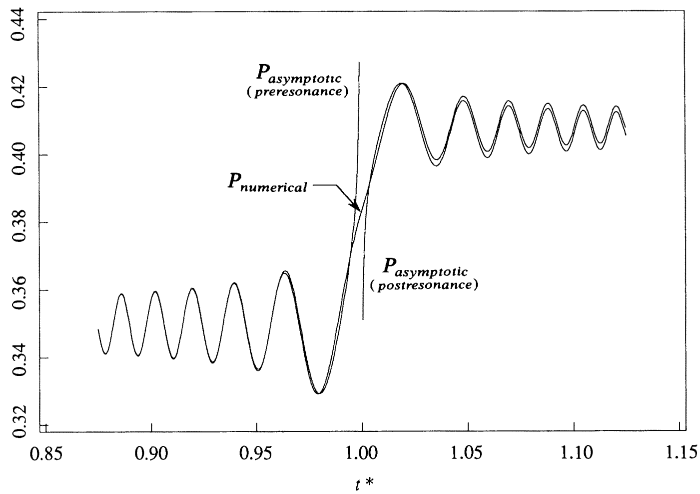

FIG. 3.3. *Subcritical asymptotic solutions using improved matching procedure.*

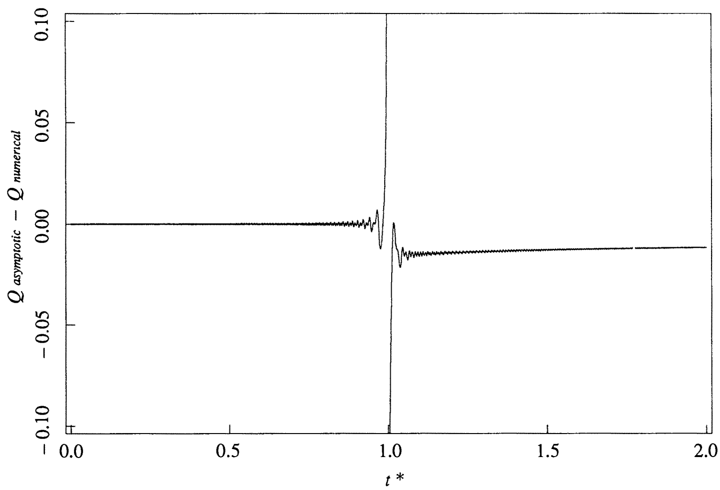

FIG. 3.4. *Subcritical phase error using improved matching procedure.*

the resonance to leading order

$$\Delta P = -\varepsilon^{(1-\alpha)/(1+\alpha)} \frac{2A(\eta_0, t_0^*)}{(1+\alpha)} \left(\frac{1+\alpha}{\sigma_\alpha}\right)^{1/(1+\alpha)} \Gamma\left(\frac{1}{1+\alpha}\right) \cos\left(\frac{\pi}{2(1+\alpha)} + \xi_0\right). \tag{3.32}$$

Note that as $\alpha \to 1$, the critical value of α, the change in the action becomes $O(1)$, as does the change in the phase constant. Therefore the solution for the interior layer (3.20) can no longer be considered valid and the case where $\alpha = 1$ must be addressed separately.

3.4. A special case, which permits an exact solution. An important special case of the reduced problem, which permits an exact solution independent of the nature of $\sigma(t^*)$, is specified by the function

$$A(P, t^*) = \gamma(t^*)\sqrt{P}, \tag{3.33}$$

which gives the equations

$$\frac{dQ}{dt} = \sigma(t^*) + \varepsilon \frac{\gamma(t^*)}{2\sqrt{P}} \sin Q, \tag{3.34a}$$

$$\frac{dP}{dt} = -\varepsilon\gamma(t^*)\sqrt{P} \cos Q. \tag{3.34b}$$

These equations are a minor extension of the reduced problem resulting from the forced oscillator discussed in § 2.2.3. (A slightly more general system corresponding to $A(P, t^*) = \gamma(t^*)\sqrt{\alpha + P}$ can be easily transformed to (3.34) by noting that a constant shift in P does not alter the Hamiltonian nature of the differential equations.) The reduced system (3.34) will be used to examine the behavior of solutions for both the critical and supercritical cases in §§ 3.5 and 3.6.

Although the coupling of equations (3.34) is nonlinear and the equations are still nonautonomous, we can solve (3.34) exactly using the transformation

$$T(t) = \gamma(t^*)\sqrt{P} \exp(iQ), \tag{3.35}$$

which converts the system (3.34) of two first-order equations to a single linear complex differential equation in T

$$\frac{dT}{dt}(t) = i\sigma(t^*)T - \frac{\varepsilon\gamma^2(t^*)}{2} + \varepsilon^2 \frac{\gamma'(t^*)}{\gamma(t^*)} T. \tag{3.36}$$

Equation (3.36) has the exact solution

$$T(t) = \gamma(t^*) \exp\left(i\frac{\tau(t^*)}{\varepsilon^2}\right)\left[C - \frac{1}{2\varepsilon}\int_0^{t^*} \gamma(s) \exp\left(-i\frac{\tau(s)}{\varepsilon^2}\right) ds\right], \tag{3.37}$$

where

$$\tau(t^*) = \int_0^{t^*} \sigma(s)\, ds, \tag{3.38}$$

$t^* = \varepsilon^2 t$, and C is a complex constant of integration found from initial conditions

$$C = \sqrt{P(0)} \exp(iQ(0)). \tag{3.39}$$

The solutions for P and Q are easily retrieved

$$P = \frac{T\bar{T}}{\gamma^2(t^*)}, \tag{3.40a}$$

$$Q = \arg(T), \tag{3.40b}$$

giving uniformly valid solutions to (3.34) over the entire domain of interest. The overbar is used here to indicate the complex conjugate. Since the integral in the solution for $T(t)$ can be evaluated by the method of stationary phase, we can match the solutions (3.40) with the outer solutions (3.7) in the pre- and postresonance regions to express the solution outside the resonance region in a more useful form. The only stationary point in the integral corresponds to the resonant time t_0^*, so in the preresonance region, for $t^* < t_0^*$, the solution for T can be asymptotically expanded from (3.37) independent of the nature of the stationary point as follows:

$$\begin{aligned} T(t) = \gamma(t^*) \exp\left(i\frac{\tau(t^*)}{\varepsilon^2}\right)\left[\sqrt{P(0)} \exp(iQ(0)) + \varepsilon\frac{i\gamma(0)}{2\sigma(0)}\right] \\ -\varepsilon\frac{i\gamma^2(t^*)}{2\sigma(t^*)} + O(\varepsilon^3). \end{aligned} \tag{3.41}$$

In the postresonance region, $t^* > t_0^*$, the order of the stationary point will modify the asymptotics of the solution; the order, $k-1$, indicates whether the resonance is critical (order of one) or supercritical (greater than one). It holds that

$$\begin{aligned} T(t) = &-\frac{\gamma(t^*)}{2\varepsilon k} \exp\left(i\frac{\tau(t^*)-\tau(t_0^*)}{\varepsilon^2}\right) \\ &\cdot\left\{\Gamma\left(\frac{1}{k}\right)\left(\frac{\varepsilon^2 k!}{|\tau^{(k)}(t_0^*)|}\right)^{1/k} \gamma(t_0^*)\right. \\ &\cdot\left[2\cos\left(\frac{\pi}{2k}\right) - \mu[1+(-1)^k]i\sin\left(\frac{\pi}{2k}\right)\right] \\ &-\mu\Gamma\left(\frac{2}{k}\right)\left(\frac{\varepsilon^2 k!}{|\tau^{(k)}(t_0^*)|}\right)^{2/k} [1-(-1)^k]i\sin\left(\frac{\pi}{k}\right) \\ &\cdot\left.\left[\gamma'(t_0^*) - \frac{2\tau^{(k+1)}(t_0^*)\gamma(t_0^*)}{k(k+1)\tau^{(k)}(t_0^*)}\right]\right\} \\ &+\gamma(t^*) \exp\left(i\frac{\tau(t^*)}{\varepsilon^2}\right)\left[\sqrt{P(0)} \exp(iQ(0)) + \varepsilon\frac{i\gamma(0)}{2\sigma(0)}\right] \\ &-\varepsilon\frac{i\gamma^2(t^*)}{2\sigma(t^*)} + O(\varepsilon^{(6-k)/k}), \end{aligned} \tag{3.42}$$

where $\mu = \operatorname{sgn}(\sigma^{(k-1)}(t_0^*))$. The higher-order terms in the stationary phase evaluation are found in Bleistein and Handelsman [3]. In the next two sections we will examine this solution for the critical and supercritical cases. In the Appendix we show that the same solution can be found by a direct transformation from the forced oscillator (2.28) to $T(t)$.

3.5. Solutions for the critical case. In the critical case, the leading-order frequency σ has the Taylor series expansion about t_0^*

$$\sigma(t^*) = \sigma_1(t^*-t_0^*) + \frac{\sigma_2}{2}(t^*-t_0^*)^2 + \cdots, \tag{3.43}$$

where $\sigma_1 \neq 0$. In the interior layer, the appropriate rescaled time is given by

$$t_1 = \frac{t^*-t_0^*}{\varepsilon} = \varepsilon t - \frac{t_0^*}{\varepsilon}, \tag{3.44}$$

where ε is the interior layer thickness in t^*. This case includes the richest $O(1)$ equation as both σ and the $O(\varepsilon)$ coupling contribute. The leading-order equations for (3.11) are then given by

$$\frac{dQ_0}{dt_1}=\sigma_1 t_1+\frac{\partial A}{\partial P}(P_0, t_0^*)\sin Q_0, \tag{3.45a}$$

$$\frac{dP_0}{dt_1}=-A(P_0, t_0^*)\cos Q_0. \tag{3.45b}$$

As mentioned, these $O(1)$ equations for the resonance region (3.45) are nonlinearly coupled, nonautonomous, and not always analytically solvable. This introduces problems not only in obtaining valid solutions during resonance but in matching the pre- and postresonance expansions (3.7). Two possibilities for obtaining this matching are (a) to match the asymptotic pre- and postresonance solutions to a numerical integration of (3.45) in the resonance region, or (b) to find a connection formula that jumps the interior layer, expressing the constants of integration in the postresonance region in terms of the constants of integration in the preresonance region. The latter can be done for certain problems (e.g., see § 3.4) but is difficult in general. The matching of a numerical interior layer also provides little advantage; it makes more sense simply to integrate the original reduced system of differential equations throughout the desired interval, and then to determine the constants in the asymptotic expansions from that numerical solution.

As discussed in § 3.4, when the function A is of the form (3.33), the differential equations have an exact solution. Using the stationary phase results (3.41) and (3.42), we evaluate P in both the pre- and postresonance regions. We then use these results to compare with a solution obtained by matching an interior layer solution with the pre- and postresonance outer solutions (3.7). In the preresonance region, we find from (3.40a) and (3.41)

$$P(t)=P(0)+\varepsilon\sqrt{P(0)}\left[\frac{\gamma(0)}{\sigma(0)}\sin Q(0)-\frac{\gamma(t^*)}{\sigma(t^*)}\sin\left(\frac{\tau(t^*)}{\varepsilon^2}+Q(0)\right)\right]+O(\varepsilon^2). \tag{3.46}$$

In the postresonance region, $t^*>t_0^*$, we find the solution for P given by (3.40a) and (3.42) using $k=2$ for the critical case where

$$\begin{aligned}P(t)=P(0)&+\frac{\gamma^2(t_0^*)\pi}{2|\sigma_1|}-\gamma(t_0^*)\sqrt{\frac{2P(0)\pi}{|\sigma_1|}}\cos\left(\frac{\tau(t_0^*)}{\varepsilon^2}+Q(0)+\frac{\mu\pi}{4}\right)\\&+\varepsilon\gamma(t_0^*)\sqrt{\frac{\pi}{2|\sigma_1|}}\left[\frac{\gamma(0)}{\sigma(0)}\sin\left(\frac{\tau(t_0^*)}{\varepsilon^2}+\frac{\mu\pi}{4}\right)\right.\\&\qquad\left.+\frac{\gamma(t^*)}{\sigma(t^*)}\sin\left(\frac{\tau(t^*)-\tau(t_0^*)}{\varepsilon^2}-\frac{\mu\pi}{4}\right)\right]\\&+\varepsilon\sqrt{P(0)}\left[\frac{\gamma(0)}{\sigma(0)}\sin Q(0)-\frac{\gamma(t^*)}{\sigma(t^*)}\sin\left(\frac{\tau(t^*)}{\varepsilon^2}+Q(0)\right)\right]+O(\varepsilon^2),\end{aligned} \tag{3.47}$$

where $\mu=\operatorname{sgn}(\sigma_1)$. Therefore from the exact solution given in § 3.4, the change in the average value of the action across the resonance is

$$\begin{aligned}\Delta P=&\frac{\gamma^2(t_0^*)\pi}{2|\sigma_1|}-\gamma(t_0^*)\sqrt{\frac{2P(0)\pi}{|\sigma_1|}}\cos\left(\frac{\tau(t_0^*)}{\varepsilon^2}+Q(0)+\frac{\mu\pi}{4}\right)\\&+\varepsilon\sqrt{\frac{\pi}{2|\sigma_1|}}\,\frac{\gamma(0)\gamma(t_0^*)}{\sigma(0)}\sin\left(\frac{\tau(t_0^*)}{\varepsilon^2}+\frac{\mu\pi}{4}\right)+O(\varepsilon^2).\end{aligned} \tag{3.48}$$

This result can also be found by matching an interior layer solution with the outer solutions in the pre- and postresonance regions (3.7). For the function A from (3.33), the $O(1)$ interior layer equations for Q_0 and P_0 corresponding to (3.45) are

$$\frac{dQ_0}{dt_1}=\sigma_1 t_1+\frac{\gamma_0}{2\sqrt{P_0}}\sin Q_0, \tag{3.49a}$$

$$\frac{dP_0}{dt_1}=-\gamma_0\sqrt{P_0}\cos Q_0, \tag{3.49b}$$

where $\gamma_0=\gamma(t_0^*)$. The transformation used to solve the problem of § 3.4 is still applicable with slight modification, giving the interior layer solutions for Q_0 and P_0

$$P_0(t_1)=\frac{T_0(t_1)\overline{T_0(t_1)}}{\gamma_0^2}, \tag{3.50a}$$

$$Q_0(t_1)=\arg(T_0(t_1)), \tag{3.50b}$$

where $T_0(t_1)$ is given by

$$T_0(t_1)=\exp\left(i\frac{\sigma_1}{2}t_1^2\right)\left[C_0-\frac{\gamma_0^2}{2}\int_0^{t_1}\exp\left(-i\frac{\sigma_1}{2}s^2\right)ds\right]. \tag{3.51}$$

Since the Fresnel integral in the solution for $T_0(t_1)$ has straightforward asymptotic expansions as $t_1\to\pm\infty$, we can easily match the solutions (3.50) with the outer solutions (3.7) in the preresonance region to find the value of C_0, then with the postresonance expansions to find the final asymptotic solutions after resonance. The asymptotic results give

$$\lim_{t_1\to\pm\infty}P_0(t_1)=\frac{\gamma_0^2\pi}{8|\sigma_1|}+\frac{C_0\overline{C_0}}{\gamma_0^2}\mp\frac{(1+\mu i)C_0+(1-\mu i)\overline{C_0}}{4}\sqrt{\frac{\pi}{|\sigma_1|}}, \tag{3.52}$$

where $\mu=\operatorname{sgn}(\sigma_1)$. Defining the constants $\delta^\pm$ and $\psi^\pm$ by

$$\delta^\pm\exp(i\psi^\pm)=C_0\mp\frac{\gamma_0^2}{4}(1-\mu i)\sqrt{\frac{\pi}{|\sigma_1|}} \tag{3.53}$$

gives the asymptotic expression for $Q_0(t_1)$ from (3.50)

$$\lim_{t_1\to\pm\infty}Q_0(t_1)=\frac{\sigma_1}{2}t_1^2+\psi^\pm, \tag{3.54}$$

where

$$\psi^\pm=\arg\left[C_0\mp\frac{\gamma_0^2}{4}(1-\mu i)\sqrt{\frac{\pi}{|\sigma_1|}}\right]. \tag{3.55}$$

Matching these interior layer expansions with the outer solution (3.7) as $t_1\to-\infty$ gives the conditions

$$p_-^{(0)}=\frac{\gamma_0^2\pi}{8|\sigma_1|}+\frac{C_0\overline{C_0}}{\gamma_0^2}+\frac{(1+\mu i)C_0+(1-\mu i)\overline{C_0}}{4}\sqrt{\frac{\pi}{|\sigma_1|}}, \tag{3.56}$$

$$\hat{\tau}_0=\arg\left[C_0+\frac{\gamma_0^2}{4}(1-\mu i)\sqrt{\frac{\pi}{|\sigma_1|}}\right], \tag{3.57}$$

where

$$\hat{\tau}_0=\tau_0^-+\frac{1}{\varepsilon^2}\int_0^{t_0^*}\sigma(t^*)\,dt^*. \tag{3.58}$$

Note that $\mu(t^*)\equiv 0$ from (3.8b) for this case. Equations (3.56) and (3.57) can be simultaneously solved for C_0

$$C_0 = -\frac{\gamma_0^2}{4}\sqrt{\frac{\pi}{|\sigma_1|}}(1-\mu i) + \gamma_0\sqrt{p_-^{(0)}}\exp(i\hat{\tau}_0). \tag{3.59}$$

We can now find the connection formula for the $O(1)$ change in the action across the resonance in terms of the preresonance constants

$$\Delta P = p_+^{(0)} - p_-^{(0)} = \frac{\gamma_0^2\pi}{2|\sigma_1|} - \gamma_0\sqrt{\frac{2\pi p_-^{(0)}}{|\sigma_1|}}\cos\left(\hat{\tau}_0 + \mu\frac{\pi}{4}\right). \tag{3.60}$$

Similarly, we find the postresonance phase constant

$$\tau_0^+ = \arg\left[-\frac{\gamma_0^2}{2}(1-\mu i)\sqrt{\frac{\pi}{|\sigma_1|}} + \gamma_0\sqrt{p_-^{(0)}}\exp(i\hat{\tau}_0)\right], \tag{3.61}$$

where the branch is easily determined.

By noting from initial conditions that

$$p_-^{(0)} = P(0) + \varepsilon\frac{\gamma(0)\sqrt{P(0)}}{\sigma(0)}\sin Q(0) + O(\varepsilon^2) \tag{3.62a}$$

and

$$\tau_0^- = Q(0) + \varepsilon\frac{\gamma(0)}{2\sqrt{P(0)}\sigma(0)}\cos Q(0) + O(\varepsilon^2), \tag{3.62b}$$

we find that the change in the average value of the action (3.60) found from matching the interior layer solution (3.50) with the pre- and postresonance expansions is the same to $O(\varepsilon)$ as ΔP given by the asymptotics of the exact solution (3.48), confirming the matching result.

Since this reduced problem corresponds to the forced oscillator discussed in § 2.2.3, we can verify the results by numerically integrating the differential equation (2.28) and comparing to the solution obtained here. For the forced oscillator (2.28),

$$\gamma(t^*) = \lambda\sqrt{\frac{1}{2(1+\sigma(t^*))}}. \tag{3.63}$$

In Fig. 3.5, we use the numerically obtained solutions for x and $\dot{x}$ to calculate the action p_1

$$p_1 = \frac{\dot{x}^2 + (1+\sigma)^2x^2}{2(1+\sigma)}, \tag{3.64}$$

then plot this function with the asymptotic outer solution (3.7b) for P, using constants found from the initial conditions $P(0)=1.0$ and $Q(0)=\pi$ for the preresonance solution, and constants found from the connection formulas (3.60) and (3.61) for the postresonance solution. Other parameters include $\varepsilon=0.05$, $\lambda=1.5$, $\theta=\pi$, and $\sigma(t^*)=-\frac{1}{2}\cos(\pi/2t^*)$.

As seen, the asymptotic solution for the reduced problem corresponds very well to the $O(1)$ behavior of the action p_1 everywhere outside the interior resonance layer. The presence of the smaller oscillations of $O(\varepsilon)$ can be accounted for by inverting the near-identity averaging transformation discussed in § 2.2.3. In Fig. 3.6, we compare the numerically obtained action with the interior layer solution found from (3.50a). Again we find excellent agreement in the interior layer.

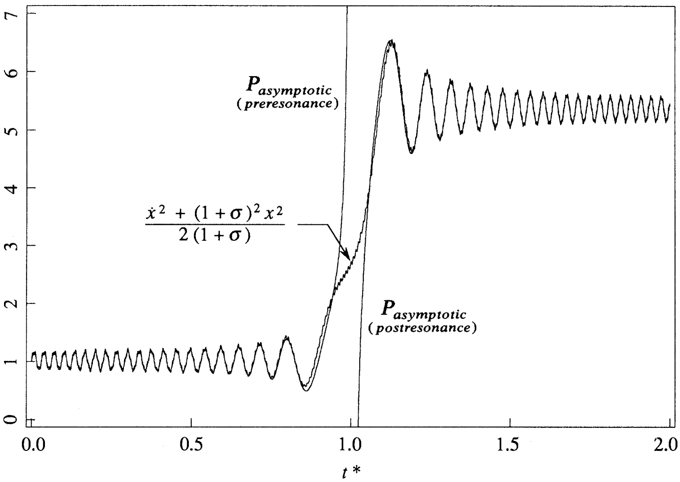

FIG. 3.5. *Asymptotic outer solutions for the critical case.*

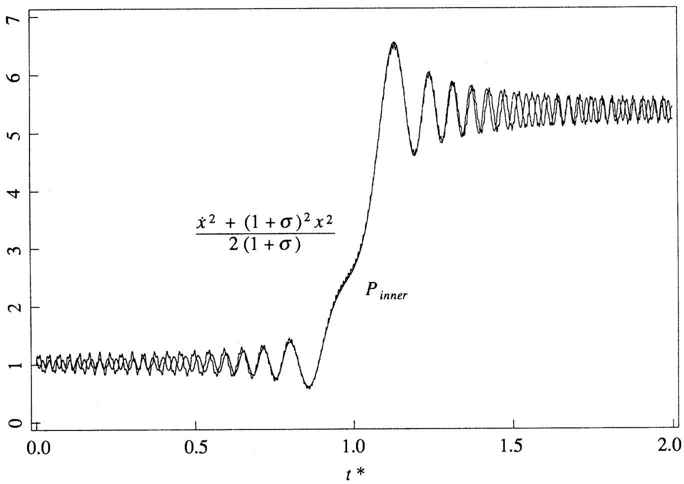

FIG. 3.6. *Interior layer solution for the critical case.*

For the general interior layer problem (3.45), similar behavior across the resonance layer is encountered for a variety of functions $A(P, t^*)$, although interior layer solutions and connection formulas are not available.

3.6. Supercritical case. The supercritical case, $\alpha > 1$, is significant as it allows an $O(1)$ change in the action across the resonance; but, unlike the case where $\alpha = 1$, which also allows an $O(1)$ change, the supercritical case has an analytical solution to the general $O(1)$ interior layer equations since these are autonomous (see [10])

$$\frac{dQ_0}{dt_1} = \frac{\partial A}{\partial P}(P_0, t_0^*) \sin Q_0, \tag{3.65a}$$

$$\frac{dP_0}{dt_1} = -A(P_0, t_0^*) \cos Q_0. \tag{3.65b}$$

This permits us to find asymptotic expansions for P and Q; however, these solutions cannot be matched with the outer expansions. Although the details are not included here (see Bosley [4]), we note that an intermediate layer solution is necessary but not available.

The exact solution found in § 3.4 can be used to examine the nature of the supercritical solutions. As expected, since the frequency remains near zero for longer times, the change in the action can be greater than $O(1)$. If we assume the leading-order frequency has an expansion near t_0^* given by

$$\sigma(t^*)=\frac{\sigma_2}{2}(t^*-t_0^*)^2+\frac{\sigma_3}{6}(t^*-t_0^*)^3+\cdots, \tag{3.66}$$

we can use the stationary phase evaluations to find the change in the action across the resonance from (3.42). The result is

$$\begin{aligned}\Delta P=&\left(\frac{6}{\varepsilon|\sigma_2|}\right)^{2/3}\frac{\gamma^2(t_0^*)\Gamma(1/3)^2}{12}\\&-\left(\frac{6}{\varepsilon|\sigma_2|}\right)^{1/3}\sqrt{\frac{P(0)}{3}}\,\gamma(t_0^*)\Gamma(1/3)\cos\left(\frac{\tau(t_0^*)}{\varepsilon^2}+Q(0)\right)\\&+\varepsilon^{1/3}\mu\left(\frac{6}{|\sigma_2|}\right)^{2/3}\sqrt{\frac{P(0)}{3}}\,\Gamma(2/3)\\&\cdot\left[\gamma'(t_0^*)-\frac{\sigma_3\gamma(t_0^*)}{6\sigma_2}\right]\sin\left(\frac{\tau(t_0^*)}{\varepsilon^2}+Q(0)\right)+O(\varepsilon^{2/3}).\end{aligned} \tag{3.67}$$

We can again verify this result by integrating the forced oscillator (2.28) for initial conditions corresponding to $Q(0)=\pi/2$ and $P(0)=1$, with the frequency

$$\sigma(t^*)=\frac{(t^*-1)^2}{8}[2+(2t^*-1)^2+5(t^*-1)^4] \tag{3.68}$$

for the value $\varepsilon=0.02$. Other parameters used are $\lambda=1.5$ and $\theta=\pi/2$. In Fig. 3.7, we show the action computed from the numerical solution and (3.64), along with the expected jump in the average value found from (3.67).

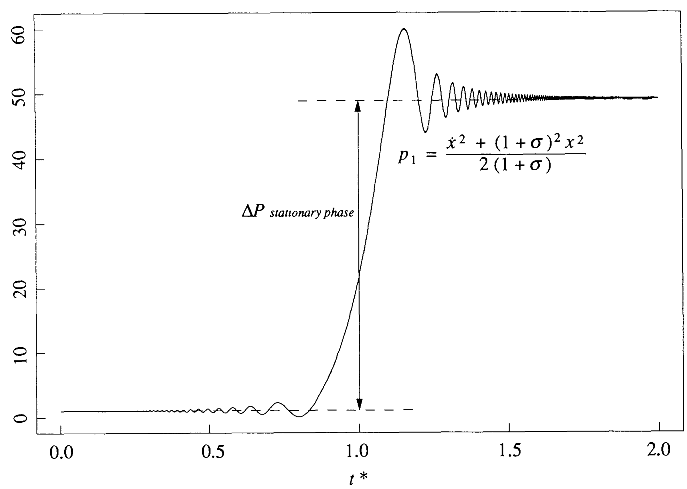

FIG. 3.7. *Change in the action for the supercritical case.*

An interesting consequence of supercritical resonance for this problem is that the leading-order postresonance action is independent of initial conditions, as is the leading-order phase

$$Q(t)=\pi+\frac{\tau(t^*)-\tau(t_0^*)}{\varepsilon^2}-\left(\frac{\varepsilon|\sigma_2|}{6}\right)^{1/3}2\sqrt{3P(0)}\frac{\sin\left(\frac{\tau(t_0^*)}{\varepsilon^2}+Q(0)\right)}{\gamma(t_0^*)\Gamma(\frac{1}{3})}+O(\varepsilon^{2/3}). \tag{3.69}$$

Thus, the amplitude and phase shift of the forcing function in (2.28) along with the specified frequency $\sigma(t^*)$ largely determine the postresonance behavior of $x(t)$, and the passage through resonance becomes much less sensitive to initial conditions than in the critical case.

Although the jump in the action is greater than $O(1)$ for the forced oscillator, the other reduced problems found in §§ 2.2.1 and 2.2.2 can only have changes in the action not exceeding $O(1)$. Since the actions are strictly positive quantities, the numerical values of the adiabatic invariants found from initial conditions put constraints on the change in the action P across the resonance.

Appendix. Solution for a resonantly forced oscillator. The forced oscillator

$$\ddot{x}+(1+\sigma(t^*))^2x=-\varepsilon\lambda\cos(t+\theta) \tag{A.1}$$

discussed in § 2.2.3 and the resulting reduced problem

$$\frac{dQ}{dt}=\sigma(t^*)+\varepsilon\frac{\lambda}{2(2P(1+\sigma))^{1/2}}\sin Q+O(\varepsilon^3), \tag{A.2a}$$

$$\frac{dP}{dt}=-\varepsilon\lambda\sqrt{\frac{P}{2(1+\sigma)}}\cos Q+O(\varepsilon^3) \tag{A.2b}$$

of which an extension is discussed in § 3.4 can be solved via the transformation

$$T(t)=\lambda\sqrt{\frac{P}{2(1+\sigma)}}\exp(iQ). \tag{A.3}$$

However, by inverting the transformations used in § 2.2.3, it is also possible to find a direct transformation from x, $\dot{x}$, and t to the complex quantity T

$$\begin{aligned}T(t)&=\lambda\exp(-i(t+\theta))\frac{\dot{x}+i(1+\sigma(t^*))x}{2(1+\sigma(t^*))}\\&\quad+\varepsilon\exp(-i2(t+\theta))\frac{i\lambda^2}{4(1+\sigma(t^*))(2+\sigma(t^*))}\\&\quad+\varepsilon^2\exp(-i(t+\theta))\frac{\lambda\sigma'(t^*)(i\dot{x}+(1+\sigma(t^*))x)}{8(1+\sigma(t^*))^3}\end{aligned} \tag{A.4}$$

to get the corresponding differential equation (3.36) for T

$$\frac{dT}{dt}(t)=i\sigma(t^*)T(t)-\varepsilon\frac{\lambda^2}{4(1+\sigma(t^*))}-\varepsilon^2\frac{\sigma'(t^*)}{2(1+\sigma(t^*))}T(t)+O(\varepsilon^3), \tag{A.5}$$

which has the solution

$$\begin{aligned}T(t)&=\frac{1}{(1+\sigma(t^*))^{1/2}}\exp\left(i\frac{\tau(t^*)}{\varepsilon^2}\right)\\&\quad\cdot\left[C-\frac{\lambda^2}{4\varepsilon}\int_0^{t^*}\frac{1}{\sqrt{1+\sigma(s)}}\exp\left(-i\frac{\tau(s)}{\varepsilon^2}\right)ds\right]+O(\varepsilon),\end{aligned} \tag{A.6}$$

where $\tau(t^*)$ is given by (3.38), $t^*=\varepsilon^2 t$, and C is a complex constant of integration found from initial conditions. The solution for x is easily retrieved from (A.4)

$$x(t)=\frac{2}{\lambda}[T_R \sin(t+\theta)+T_I \cos(t+\theta)] \\ -\varepsilon\frac{\lambda\cos(t+\theta)}{2(1+\sigma(t^*))(2+\sigma(t^*))}+O(\varepsilon^2), \tag{A.7}$$

where T_R and T_I are the real and imaginary parts of $T(t)$, respectively.

The transformation (A.4) has a natural extension to the same forced oscillator with an $O(1)$ forcing term

$$\ddot{x}+(1+\sigma(\tilde{t}))^2 x=-\alpha\cos(t+\theta), \tag{A.8}$$

where $\tilde{t}=\varepsilon t$ and α is no longer restricted as small. This is the problem discussed by Kevorkian in [9]. The direct transformation is now given by

$$\begin{aligned} T(t)&=\alpha\exp(-i(t+\theta))\frac{\dot{x}+i(1+\sigma(\tilde{t}))x}{2(1+\sigma(\tilde{t}))} \\ &+\exp(-i2(t+\theta))\frac{i\alpha^2}{4(1+\sigma(\tilde{t}))(2+\sigma(\tilde{t}))} \\ &+\varepsilon\exp(-i(t+\theta))\frac{\alpha\sigma'(\tilde{t})(i\dot{x}+(1+\sigma(\tilde{t}))x)}{8(1+\sigma(\tilde{t}))^3} \\ &+\varepsilon\exp(-i2(t+\theta))\frac{\alpha^2\sigma'(\tilde{t})(12+18\sigma(\tilde{t})+7\sigma^2(\tilde{t}))}{16(1+\sigma(\tilde{t}))^3(2+\sigma(\tilde{t}))^3}, \end{aligned} \tag{A.9}$$

which gives the differential equation for T

$$\frac{dT}{dt}(t)=i\sigma(\tilde{t})T(t)-\frac{\alpha^2}{4(1+\sigma(\tilde{t}))} \\ -\varepsilon\frac{\sigma'(\tilde{t})}{2(1+\sigma(\tilde{t}))}T(t)-\varepsilon\frac{i\alpha^2\sigma'(\tilde{t})}{16(1+\sigma(\tilde{t}))^3}+O(\varepsilon^2) \tag{A.10}$$

and has solution

$$T(t)=\frac{1}{(1+\sigma(\tilde{t}))^{1/2}}\exp\left(i\frac{\tau(\tilde{t})}{\varepsilon}\right) \\ \cdot\left[C-\frac{\alpha^2}{4\varepsilon}\int_0^{\tilde{t}}\frac{1}{(1+\sigma(s))^{1/2}}\left(1+\frac{\varepsilon i\sigma'(s)}{4(1+\sigma(s))^2}\right)\exp\left(-i\frac{\tau(s)}{\varepsilon}\right)ds\right]+O(\varepsilon). \tag{A.11}$$

Inverting (A.9) to find the solution for $x(t)$ to $O(\varepsilon)$ gives

$$\begin{aligned} x(t)&=\frac{2}{\alpha}[T_R\sin(t+\theta)+T_I\cos(t+\theta)]-\frac{\alpha\cos(t+\theta)}{2(1+\sigma(\tilde{t}))(2+\sigma(\tilde{t}))} \\ &+\varepsilon\sigma'(\tilde{t})\left[\frac{T_I\sin(t+\theta)-T_R\cos(t+\theta)}{2\alpha(1+\sigma(\tilde{t}))^2}\right. \\ &\left.+\frac{\alpha\sin(t+\theta)(8+11\sigma(\tilde{t})+4\sigma^2(\tilde{t}))}{4(1+\sigma(\tilde{t}))^3(2+\sigma(\tilde{t}))^3}\right]. \end{aligned} \tag{A.12}$$

This solution is equivalent to that found in Gautesen [6]. The integral in the solution for $T(t)$ again can be evaluated by the method of stationary phase, to obtain a more useful form to the desired order.

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