The Canard Unchained *or* How Fast/Slow Dynamical Systems Bifurcate

Marc Diener

Dedicated to Professor Georges H. Reeb

Canards are a new phenomenon in differential equations, arising from the application of nonstandard analysis to perturbation theory. The origin of canards goes back to the time when, in Oran, we studied [2] the evolution of the amplitude of the periodic solution of the equation

$$\epsilon\ddot{x} + (x^2 - 1)\dot{x} + x = a$$

with respect to the value of the parameter a. From a first (elementary) study, we deduced that we could expect, on microcomputing these solutions, that the choice $\epsilon = 0.01$ for the constant ϵ would make this amplitude change very fast for values of a near 1. But the results exceeded our expectations: At one critical value $\bar{a} = 0.9987404512$ a change of 10^{-10} of the parameter's value (which was the smallest change that could be taken into account by our computer) produced an amplitude drop of about 80%. It was as if the existence of medium size solutions would be a "canard"[1]!

Canard is now the name of a type of solution of a slow-fast differential system, to which the above "missing" medium-size solutions belong, that had previously been ignored. These solutions are at one and the same time exceptional, and unavoidable in the bifurcations of slow-fast systems. As will be seen, the existence of canards permits an explanation of the roughness of behaviour changes in solutions, that were traditionally felt to be discontinuities or "catastrophes". In addition, it allows us to describe the system's behaviour during the change, as much from the qualitative point of view, as from the analytical one.

A heuristic definition of canards is easy to understand with the help of some basic notions on slow-fast vector fields. By means of nonstandard analysis, it can immediately be turned into a rigorous one. It is then possible to state and prove the above mentioned properties, in words that can comfortably be interpreted heuristically by those who are not familiar with this modern theory of infinitesimals.

Slow-Fast vector fields

A slow-fast system is, in a general way, a differential "problem", such as differential equation or a vector field. Its solutions behave in two alternative ways:

[1] Canard = false report, from the old-French "vendre un canard à moitié" But see also figure 10 for the true origin of the name.

The non-standard duck-hunters, during one of their weekly brain-storming sessions *(from the left to the right: I. P. van den Berg, E. Benoit, J-L. Callot, the author (on the floor), and F. Diener).*

 © 1984 Springer-Verlag New York

There are periods during which the solution moves slowly, but these periods are separated by small time intervals in which the solution "jumps" from one value to another. The most famous slow-fast system is Van der Pol's equation

$$\epsilon \ddot{x} + (x^2 - 1)\,\dot{x} + x = 0,$$

which, among other things, governs the flow of electric current through triodes. More recently, slow-fast systems have been applied to mathematical modelling of biological processes, such as the systole/diastole slow-fast behaviour of the cardiac muscle [17]. They have also been used in a theoretical approach to the phenomena of "strange attractors" [11], [7]. Slow-fast behaviour arises when some parameter takes relatively small values compared to other parameters of the system—for example, the network's capacitance or the concentration of one chemical ion. This small constant will always be denoted by ϵ.

Locally, a slow-fast system can be written in the form of a *slow-fast vector field*

$$\dot{x} = f(x,y)$$
$$\dot{y} = \frac{1}{\epsilon}\, g(x,y) \qquad \epsilon \text{ small and positive,}$$

where f and g are smooth functions, for the moment assumed to be standard (that is, not nonstandard!). For full generality, x and y should belong to $\mathbf{R}^p$ and $\mathbf{R}^q$, but we shall restrict ourselves to the case $p = q = 1$[2]. One important feature for the understanding of the behaviour of the solutions of such a vector field is its slow-curve ζ, defined by $g(x,y) = 0$: Except in its "very near vicinity" (defined below as the *halo* of ζ), y varies rapidly. For, since ϵ is small, $\dot{y} = g(x,y)/\epsilon$ is large: So the solution jumps almost vertically towards (the halo of) a point A on the slow curve, where $g'(A) < 0$. Such a point is therefore called "attracting"[3] (see figure 1).

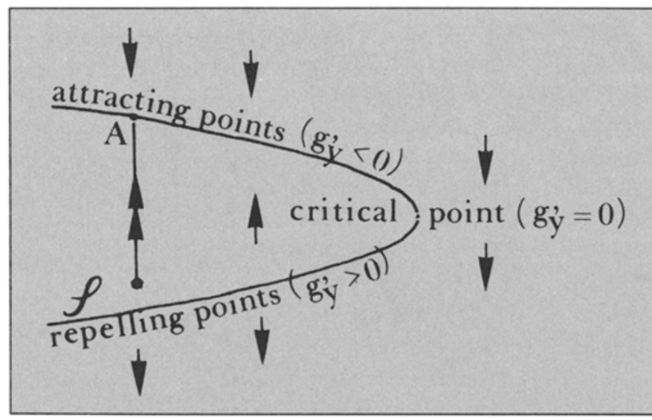


Figure 1. Attracting, repelling, and critical points of the slow-curve ζ of a slow-fast vector field.

Rivers. This characteristic feature of slow-fast vector fields suggests a new way to apply them: namely, to describe, explain, and compute the "rivers" of (ordinary) vector fields. This is a phenomenon that is often observed on phase portraits of non-linear vector fields. It usually seems to be connected with the growth of the exponential function (see figure 2); it distinguishes those trajectories that grow in a more reasonable way. A river is clearly discernible in figure 3a [4], which is the phase-portrait of $\ddot{x} + \dot{x}^2 + x = 0$. One trajectory, here a parabola, canalizes the flow on the left side on the figure. By looking at such trajectories with a "macroscope", that is, by diminishing the units in a suitable way (here letting $X = \epsilon x$, $Y = \sqrt{\epsilon}\dot{x}$) in order to see large pieces of trajectories, one observes the trajectories of the field associated with $\epsilon X'' + X'^2 + X = 0$ (see figure 3b). This slow-fast vector field gives a good caricature on the phenomenon observed in the first field. At this scale, it can

[2] For the case $p = 2$ as applied to chaotic behaviour, see [1]

[3] Trajectories may enter the halo of ζ (see § halos and galaxies) only close to points of ζ where $g'_y \leqslant 0$. Indeed, when entering $hal(\zeta)$, g^2 decreases, thus $0 \geqslant d(g^2)/dt = 2g\dot{g} = 2g(g'_x f + g'_y g/\epsilon) \simeq 2g'_y g^2/\epsilon$. The last equivalence holds for a solution, as long as $g \not\simeq 0$. Thus, by Fehrele's principle (see § permanence principles), it carries over to some point where $g \simeq 0$. In the same way, trajectories may leave $hal(\zeta)$ only close to points of ζ where $g'_y \geqslant 0$. The points of ζ are called attracting, repelling, or critical, according as g'_y is negative, positive, or zero.

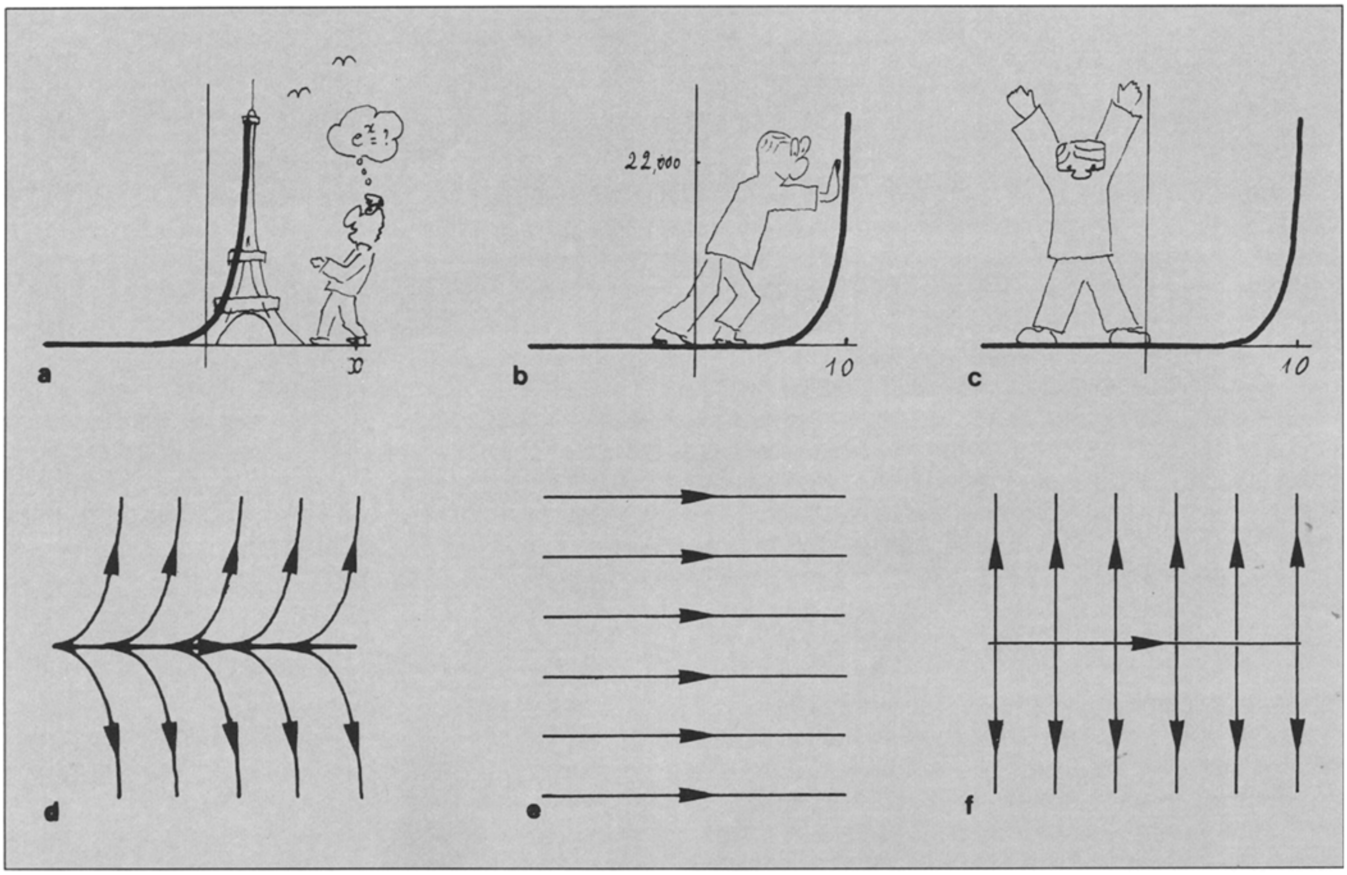


Figure 2. Pranks of the exponential function. Did you ever try to illustrate to somebody, with an exact diagram, that "the exponential function e^x is positive and defined for all x"? Perhaps you first plot its graph, choosing the same unit of both axes (figure a). Sorry not to be able to plot, say, e^{10} ($\approx$22,000), you certainly will think of diminishing the units on the y-axis . . . which of course will just push your graph to the right! Now, who do you expect to believe that e^x is non-zero for all x? Your problem will be the same if you want to illustrate, on the simplest possible ordinary linear differential equation, namely $y' = y$, that for any given $(x^\circ, y^\circ) \in \mathbf{R}^2$, there exists one and only one solution $y(x)$ such that $y(x^\circ) = y^\circ$, and that it is defined for all x. Here, it is no use to try a unit-change on the y-axis: If you want to plot ten or a dozen well-spaced solutions, you will inevitably get the drawing (d). With such a plot before the eyes of your interlocutors, what answer can you expect to the question: "If plotted on a piece of rubber, picture (d) can be deformed in picture (e) or (f): *which* one? (after J. L. Callot, Thesis, Strasbourg, 1981).

also be seen that the "river" is a canard, in a sense we can now make precise.

Canards. Once a solution has reached a point close to an attracting point of the slow-curve ζ, it necessarily stays inside the halo of ζ. Here, x may vary by an "appreciable" amount (i.e., neither small, nor large), in its slow motion (since $\dot{x} = f(x,y)$), and y must vary in a way which is *bound* to x, in the sense that $g(x,y)$ has to stay small.[4] This motion along the slow-curve proceeds until the solution gets close to a critical point C of ζ; that is, a point where g'_y vanishes. Then, usually, the solution leaves the halo of ζ, and jumps towards a new attracting point of ζ (see figure 4a). If on the contrary, it continues to follow the slow-curve along its "repelling" part where $g'_y > 0$, we shall call it a *canard* at C (see figure 4b, c, d). So, the river of $\epsilon X'' + X'^2 + X = 0$ really is a canard.

Bifurcations, the Biotope of Canards. It follows from the definition of canards, that if a vector field exhibits a canard at C, then its slow-curve must be in a "canard-configuration" at C. That is, C must be a critical point lying between an attracting and a repelling segment; or, in case of a canard with a jump, lying on a vertical line through another critical point. Figure 4 shows the three least degenerate of these slow-curve configurations. In spite of this, only the first one (fold-point) is robust. Both of the other cases can be destroyed by a perturbation of g (see figure 5). And for the fold-point, it is easy to understand that, in order to follow the slow-curve from one side of the critical point to the other, the x-motion has to reverse its di-

[4] If f and g are regular enough, for example C^1, the speed of y also becomes and stays slow, so that not only g, but also $dg/dt(=g'_x\dot{x} + g'_y\dot{y})$ stays small. But otherwise, the solution may "sputter", that is show small jerks [15]: if it does so, the motion is only "macroscopically" slow.

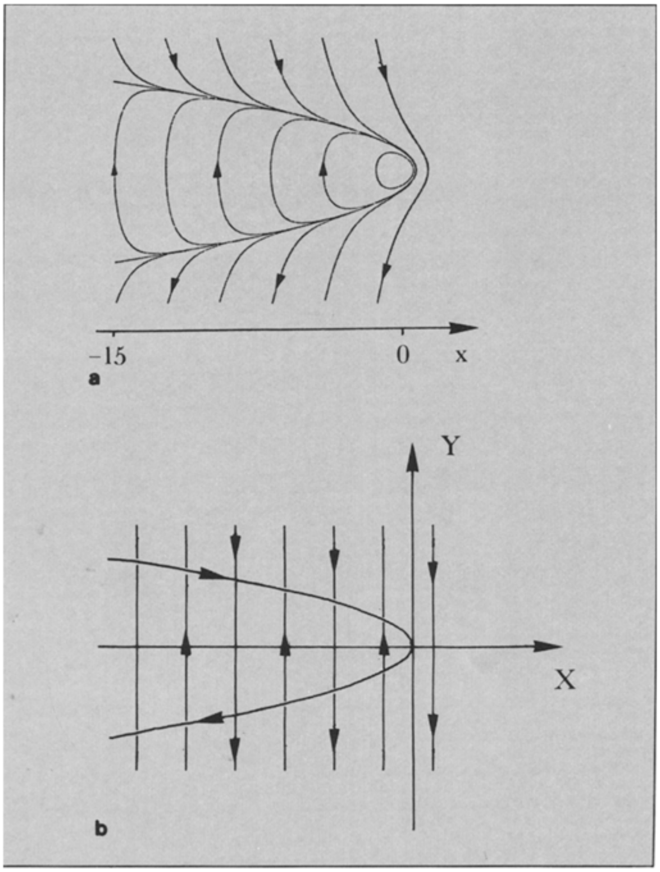


Figure 3. River, and river under a macrosope. (a) Trajectories of the (ordinary) vector field $\dot{x} = y$, $\dot{y} = -x - y^2$, associated with $\ddot{x} + \dot{x}^2 + x = 0$. The parabolic trajectory $y^2 + x - 1/2 = 0$ canalizes, in the left part of the figure, the other trajectories (of equation $y^2 + x - 1/2 = e^{-2x}$). In its upper part, like a river, it drains the trajectories of the vector field's flow, that come and intermingle with it on the plot. On its lower part, the trajectories leave it again; the parabolic trajectory behaving there as an irrigation channel. The starboard-leaving trajectories flow into the attractive part of the river, closing up and forming periodic solutions. (b) The same phase-portrait, through the macroscope $X = \epsilon x$, $Y = \sqrt{\epsilon} y$, where the vector field becomes $\dot{X} = Y$, $\dot{Y} = (-X - Y^2)/\epsilon$. Letting $T = \sqrt{\epsilon}\, t$, and $' = d/dT$, the original equation becomes the slow-fast $\epsilon X'' + X'^2 + X = 0$.

Macroscope (or megaloscope): optical instrument reducing an elephant to the size of a mouse (after Lewis Carrol, mentioned by A. Troesch [14]).

rection close to the critical point, and thus $f(C) = 0$. So we can see that f, g, and g'_y vanish at the fold-point C. This can *also* (see figure 6) be destroyed by a perturbation! This (structural) instability explains why canard-configurations were omitted from classical theories of slow-fast vector fields [9], [16].

But on the other hand, each of these three singular situations is (under some trivial "transversality" conditions) of "codimension one". For each of them, there exists a single-parameter family of slow-fast vector fields, say $\dot{x} = F(x,y,a)$, $\dot{y} = G(x,y,a)/\epsilon$ which "unfolds" the singularity. This means that for $a = 0$ it is equal to the singular configuration; and it will be equivalent

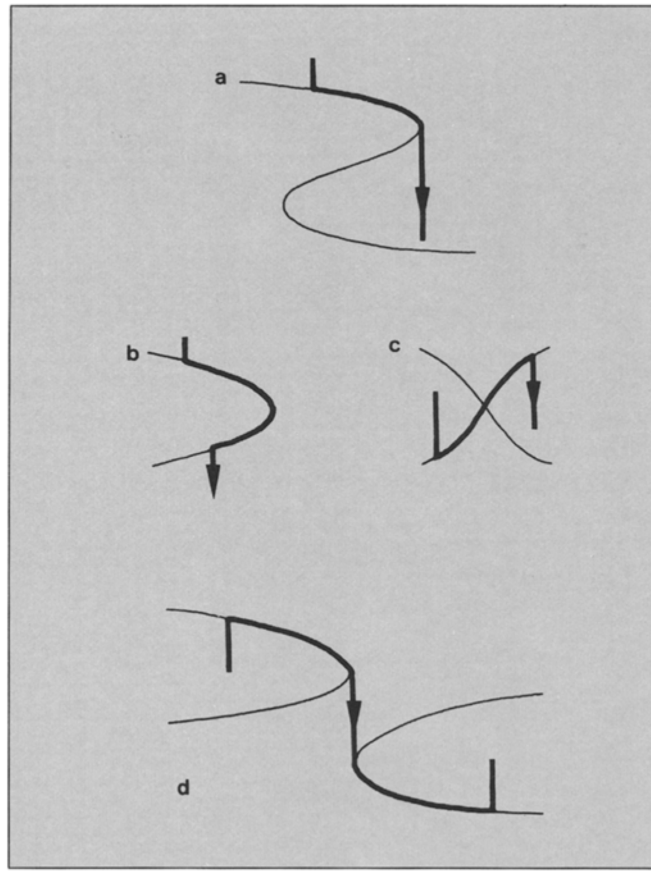


Figure 4.
a) usual behavior
b) canard at a fold-point
c) canard at a Morse-point
d) canard with jump

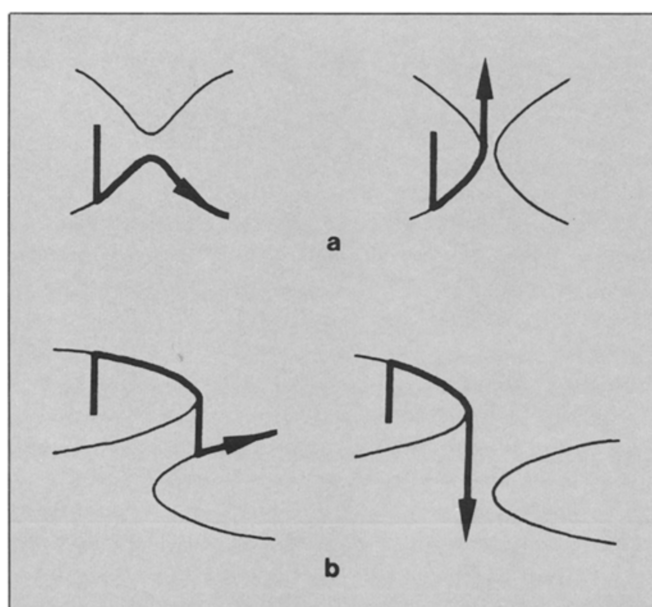
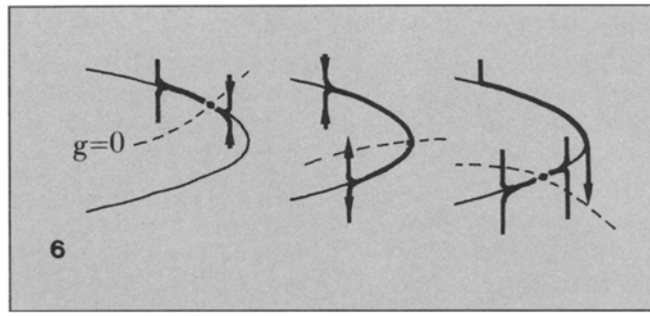


Figure 5.
a) deformation of a Morse-point canard configuration
b) deformation of a canard with jump from fold-point to fold-point configuration

to any of the different ways that the field may degenerate locally, for some value of a near zero. Moreover, this family may itself be chosen to be stable; that is any family $(\hat{F},\hat{G})$ near to (F,G) will exhibit the same configurations as (F,G). In other words, canard-configurations have to be taken into account once one is interested in (one or more parameter) *families* of slow-fast vector fields. They form the intermediate (or bifurcation) configurations between two stable one.

Thus the importance of canard-solutions (and now, not just the necessary "canard-configurations") resides in the following fact. Given any parameter-dependent family of slow-fast vector fields, which $a = 0$ is equal to one of the above canard-configurations, and which unfolds it, there exists a "canard-value" of a close to 0. That is, a value of a for which the vector field exhibits a canard [5]. Thus canards are privileged observers of major changes in the solutions during a bifurcation of the vector field.



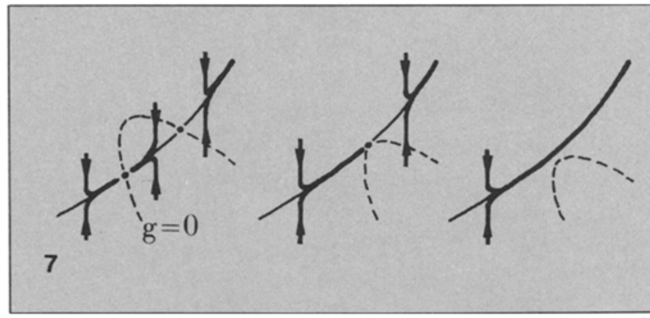


Figure 6. Perturbation of a slow-fast vector field with a canard at a fold-point: The fold-point is necessarily close to a rest-point in order to allow the canard to reverse its x-motion.

Figure 7. The saddle-node bifurcation: It is the unique generic slow-fast vector field bifurcation that does not happen via canards.

A Trip Through Non-Standard Analysis

In order to relate, in mathematically suitable words, what we may learn from these observers, a small linguistic journey will be useful, through a region which is still considered to be new, even though it had been re-discovered during the sixties [12]: *non-standard analysis*. It will also allow us to acquire some attitudes of mind which are needed there, and which give a perfectly precise meaning to what has been written above. That exposition, which was made as heuristic as possible, was conceived in order to permit a precise and correct mathematical interpretation once the adjective "small" had been defined (likewise "very near vicinity", "near", "close", and "follow" which can immediately be derived from it in a metric space).

Small and Limited. The physicist or the chemist usually has no trouble with the adjectives "small" or "large". Even if he wishes to use the means that mathematics places at his disposal, he will usually be able to say whether the mass of such and such particle, or the concentration of such and such ion, may be considered to be large or small relative to some other quantity. To the mathematician, on the other hand, the question of orders of magnitude is still a difficult one. The solution which is generally adopted is to consider the small parameter as a variable that tends to zero. But to treat a quantity which is a constant as variable cannot be more than a makeshift, introducing very unfortunately one more variable in problems where so many things are already changing, and where fixed quantities are all the more welcome.

Non-standard analysis deals with that question by introducing one new predicate: the adjective "standard". A set, a function, a number may be standard or non-standard. Every uniquely defined object of classical mathematics will be standard. A real number will be said to be *small* (or to be an infinitesimal) if its absolute value is smaller than any positive standard real number.

One postulates, among other things, that any infinite standard set contains non-standard elements. Doing so, will not introduce any contradiction in mathematics, nor will this new postulate enable us to prove new standard theorems . . . from a purely logical point of view. Of course, it is quite different from the practical point of view! The postulate shows that there exist nonstandard integers, which have to be "large", that is larger than any standard integer. By taking the inverse of such large integers, one gets small, non-zero, real numbers. A real number is said to be *limited* if its absolute value is smaller than some standard integer.

Paradoxes, or External Sets. Beside the obvious advantages that arise from a true infinitesimal calculus, non-standard analysis provides a precious tool: the so-called *external sets*. One may say that the paradoxes, pointed out by G. Berkeley after Leibnitz had introduced his infinitesimal calculus, were built up of just these external sets. Take for example the set of all integers n such that $n\epsilon$ is small (ϵ being some non-zero infinitesimal). Clearly 0 belongs to this set, and if n is in it, so is $n + 1$; however, though satisfying the induction principle, this set is not $\mathbf{N}$, for $\omega = [1/\epsilon] + 1$

does not belong to it. Other examples of external sets frequently arising are the "principal galaxy" **G**, set up of all limited real numbers, and the "halo of zero" m which collects together all infinitesimals.[5]

External sets are built up from standard sets such as **N**, **R**, or $\mathcal{C}^\infty(0,1)$, and from internal sets such as $(0,\omega)$, $\mathcal{C}^1(0,\epsilon)$, $\mathbf{Z}/\omega\mathbf{Z}$ with help of the new (or "external") predicate "standard", or one of those derived from it, such as "small", "large", or "limited". They become paradoxical only when one trys to apply classical theorems to them. But of course these were proved only for sets which are "internal" to the classical theory, that is, built up by the classical rules: set of subsets, powerset, or subsets defined by using the "internal" relation $\in$, or a relation derived from it, like $\subset$, $<$, $\leqslant$. A subset will be called *external* as soon as at least one classical theorem does not apply to it. So **G** and m are external sets, and so is our example of a bounded subset of **R** with no least upper bound.

Halos and Galaxies. Those external sets which appear spontaneously when studying slow-fast vector fields are of two kinds [3]. On the one hand we have "halos", which may be defined as the pre-image of m by an internal function. So for example, $hal(\zeta)$, the "halo of the slow-curve", consists of all (x,y) such that $g(x,y)$ is small. On the other hand there are "galaxies", which are the pre-image of **G** by an internal function. As is easily seen, the complement of a halo in an internal set is a galaxy, and conversely. In the box we give some further useful halos and galaxies.

Fehrele's Principle. From the definition, classical theorems *a priori* do not apply to external sets. In order to make these sets be more than merely a convenient way of speaking, one must prove results about them: Among these theorems are the "permanence principles", one being "Fehrele's principle". It states that no halo is a galaxy. In other words, if a halo contains a galaxy, it must overflow it. Here is an example about slow-fast vector fields. Consider some trajectory through some limited point $(x_\circ,y_\circ)$ which does not belong to $hal(\zeta)$. Since dx/dy $(=\epsilon f/g)$ is small, then as long as the trajectory stays inside the galaxy of limited points that are outside the halo of the slow-curve, it stays close to the vertical line through its "initial condition" $(x_\circ,y_\circ)$. Now, by Fehrele's principle, this last property will still be true for the halo of *one* point of the slow-curve (or of large ordinate). This leads to one of the fundamental observations about slow-fast vector fields: "Any trajectory is almost vertical until it reaches some point in the halo of an (attracting or critical) point

Some Halos and Galaxies in R

- The galaxy $\underline{\mathbf{N}}$ of all standard integers
- The galaxy $\overline{\mathbf{G}}$ of all limited real numbers. It is also called the "principal galaxy".
- The halo m of all infinitesimals. It is also called the halo of zero.
- The galaxy $\epsilon\mathbf{G} = \{\epsilon\, x \mid x \in \mathbf{G}\}$ and the halo $\epsilon m = \{\epsilon\, x \mid x \in m\}$. They are called the ϵ-galaxy of zero and the ϵ-halo of zero.
- For any positive infinitesimal ϵ, the ϵ-microhalo, which consists of all real numbers with absolute value less than ϵ^n for any standard integer n.
- For any positive infinitesimal ϵ, the ϵ-microgalaxy, which consists of all real numbers with absolute value less than $e^{1/\omega\epsilon}$ for any large integer ω.

Notice that all ϵ-galaxies are homothetic to each other, and so are the ϵ-halos. On the other hand, for any non-zero x in the ϵ-microgalaxy, there exists x' also in it, such that x'/x is large. The same is true for the ϵ-microhalo.

A of the slow-curve", which will be stated, for short, as "the trajectory reaches $hal(\zeta)$ at A". Reversing time, we get a precise meaning for "the trajectory leaves $hal(\zeta)$ at R". Let us just mention one more observation derived from Fehrele's prinicple: Once a (positive half-) trajectory has reached $hal(\zeta)$ at A, it necessarily follows ζ until some critical point, or rest point, or large point (that is a point with at least one large coordinate).

Analytical Properties of Canards

A good many of slow-fast vector field bifurcations are necessarily associated with the existence of canards. This results from the fact that, since ϵ is non-zero, solutions must depend continuously on the parameter. As one may understand from figure 8, canards are the necessary intermediaries between the phase-portraits before and after the bifurcation. This will give us information on what happens when such a bifurcation takes place, both from the qualitative and the quantitative point of view—what we shall call "analytical" properties.

In order to make things precise, let us assume that $\bar{a}$ is a near-standard canard-value of the parameter for the slow-fast vector field

$$\dot{x} = F(x,y,a),\ \dot{y} = G(x,y,a)/\epsilon,$$

where F and G are smooth standard functions. Let $a_\circ$ be the standard part of $\bar{a}$, that is, the unique standard real number near to $\bar{a}$. Finally, let $f(x,y) = F(x,y,a_\circ)$ and $g(x,y) = G(x,y,a_\circ)$; then the slow-curve ζ has equation $g(x,y) = 0$.

[5] m as "monad", according to Leibniz's terminology. Notice that m is the unique maximal idea of the ring **G**. The quotient field $\mathbf{G}/m$ is isomorphic to the external set of all standard real numbers. It can be shown [3], that this external set is neither a halo, nor a galaxy.

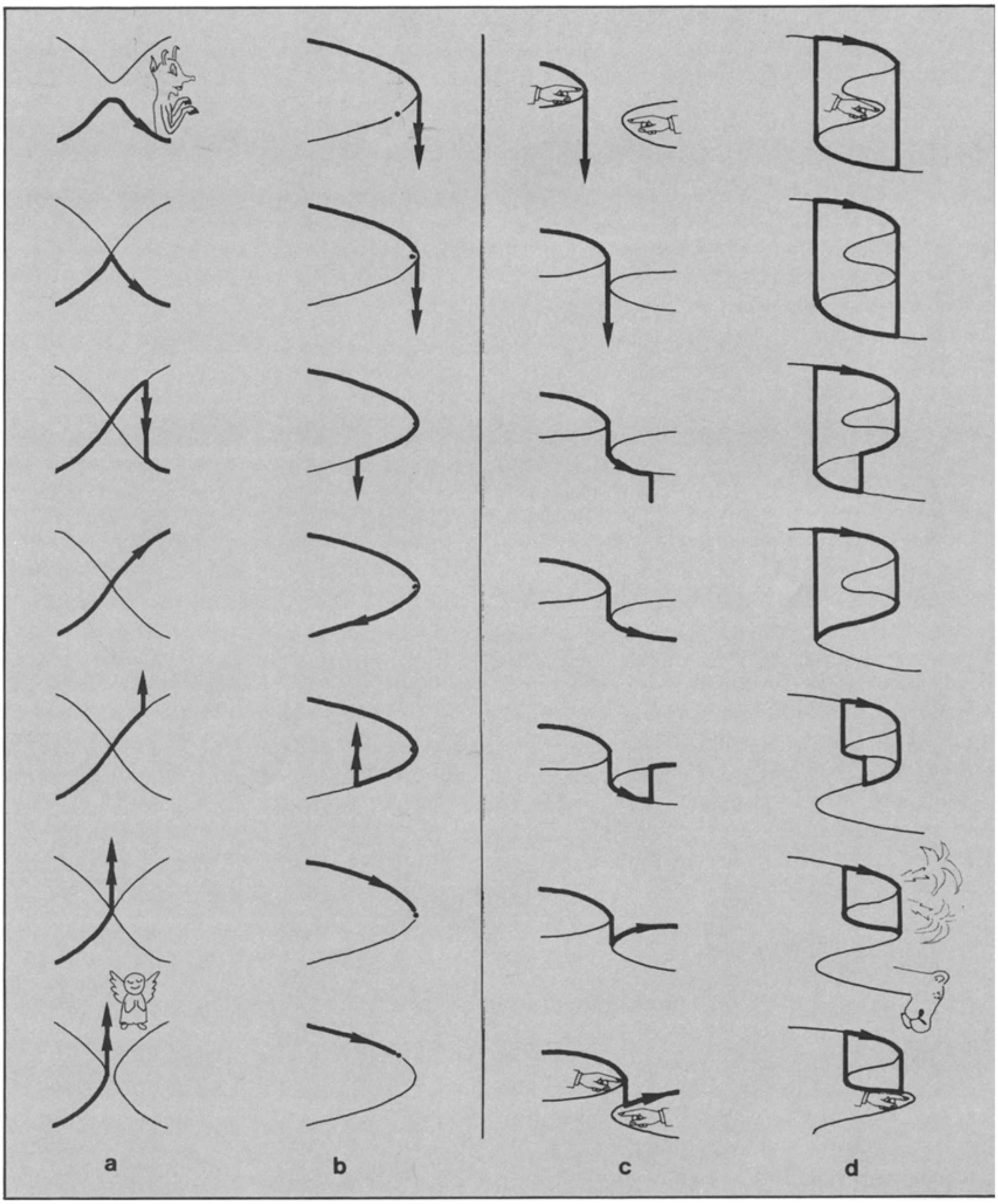


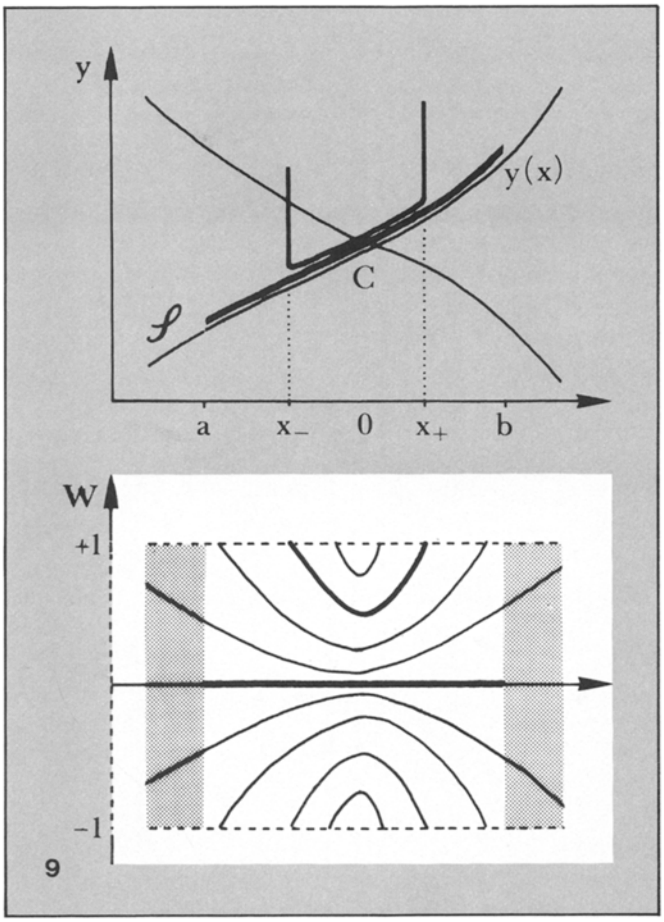
Figure 8. Various bifurcations associated with canards. (a) The Hell-and-Heaven bifurcation of a Morse-point. (b) The crossing of a fold-point by a node-rest-point: This produces also a Hopf bifurcation (see below); the magnifying glass $Y = g(x, y)/\epsilon$ reduces (b) to (a). (c) A bifurcation associated with jumping-canards. (d) This bifurcation of "the kneeling down camel" gives an example of a "catastrophic" expansion of a cycle's size, without any topological change of the phase-portrait.

The Way-In Way-Out Function. Now we have caught a canard that follows the slow-curve from the attracting side of the critical point, say (0,0), to the other repelling side. It will enable us to determine the behaviour of all other trajectories that reach $hal(\zeta)$ in the part that contains the canard. And it lets us know that there exists a relation, that can be computed, between the entering and the leaving point. As an example, here is the relation in the case of a (non-resting) Morse-point (see figure 9): The entering-point's and the leaving-point's abscissa x_- and x_+ will give the same value to the function

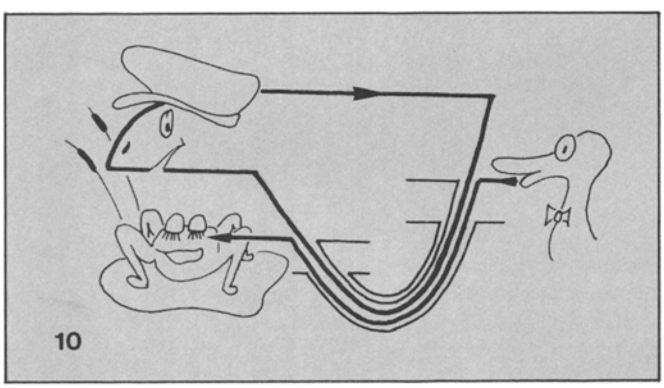
$$\phi(x) = \int_0^x (g'_y/f)\,(s, y(s))\,ds$$

where $s \mapsto y(s)$ is the function whose graph is the slow-curve segment followed by the canard. So, a canard tells us to which leaving trajectory-segment any entering one connects (see figure 10).

This disentanglement of the trajectories' behav-



9



10

Figure 9. The way-in way-out function and the singular equation of variation associated with a canard following ζ between points with abscissas a and b.
Figure 10. ". . . a canard tells us to which leaving trajectory-segment any entering one is connected."

iour inside $hal(\zeta)$ can be obtained by a crafty "microscope", $W = (y - y_\circ)^{[\epsilon]}$ introduced in that problem by Benoit [2]. It straightens out the canard-trajectory $y = y_\circ(x)$ on the x-axis $W = 0$, and stretches out $hal(\zeta)$ to the halo $[a,b] \times H$, where $H = \{\eta^{[\epsilon]}, \eta \simeq 0\}$ (see figure 9). I will not give here the complete argument, but I just want to point out that, if one knows *a priori* that some solution stays close to the canard, then it is easy to see that through the microscope it will stay close to some solution of

$$dW = (g_y'/f)\ (x,y_\circ(x)) \cdot W$$

By analogy to the classical "equation of variation" which arises when looking at any solution of a standard vector field through a "magnifying glass" $Y = (y - y_\circ)/\epsilon$, I would call this equation the "singular equation of variations". Now, this is what happens for a canard at Morse-point; but, as was pointed out in the caption to figure 8, a canard at a fold-point reduces to the Morse-point case. Thus one may also form a singular equation of variation for this case, and get the way-in way-out function for a canard at a fold point.

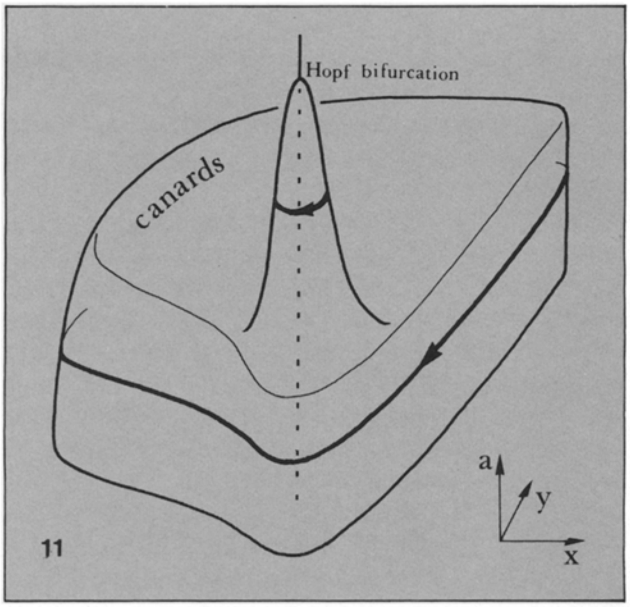
A Canard's Life is Short. Let us choose a value $\bar{a}$ of the parameter such that the vector field has the "longest possible canard" at C, that is a canard following the slow-curve on both sides of the critical point C, until it meets a new critical point, or a rest point, or a point with large norm. By continuity arguments, such an $\bar{a}$ necessarily exists and such a value may be considered as the midpoint of the process of change that accompanies the bifurcation. By a clever modification [2] of the previous microscope, Callot succeeded in measuring how a change in the value of a reduces the size of the canard. It is thus possible to determine in particular the discrepancy between two canard-values. The result is spectacular: This discrepancy is less than ϵ^n for any limited n; more precisely, the set of all those discrepancies is equal to the ϵ-microgalaxy. The "catastrophe" accompanying the bifurcation is much sharper than one would expect: The effect of the smallness of ϵ is "amplified" by the property of colossal growth of the exponential function (see figure 2).[6]

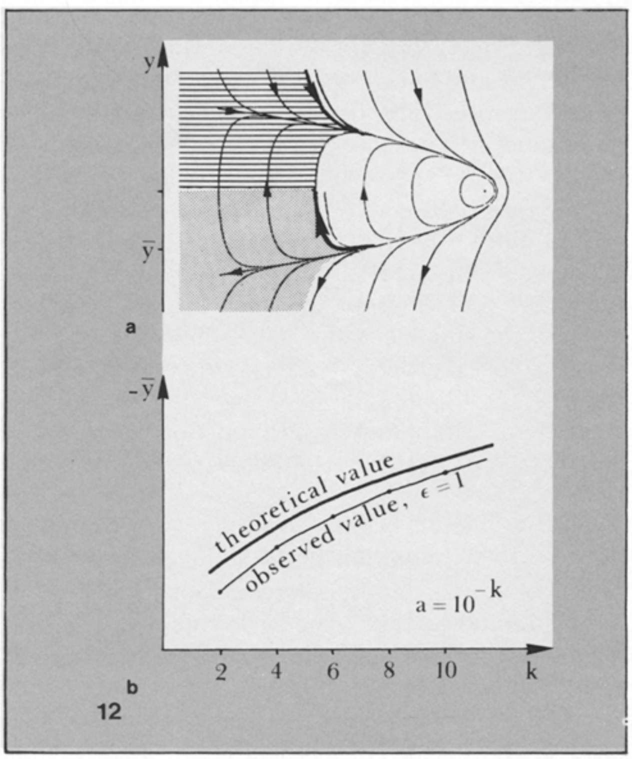
On figure 11 are drawn, as a function of a, the source and attractors (rest-point or cycle) of the equation

$$\epsilon\ddot{x} + (x^2 - 1)\dot{x} + x = a$$

(for $\epsilon = 0.1$, that is ten times as much as in the first instance above). Notice first that for $a = 1$, a Hopf bifurcation takes place: The cycle collapses and "swallows" the unstable rest-point, after which only a stable rest-point is left (top of picture). But before this, for a about 0.99875, another phenomenon takes place, much sharper, which is a rapid drop in the cycle's size. The cycles of medium size are precisely the canards that necessarily appear in any Hopf bifurcation [5]. Notice that, even when close together, canards and Hopf bifurcation are two distinct phenomena, taking place for distinct values of the parameter. F. Diener has shown [4] that, in analogy with the Hopf bifurcation for which it is possible to compute the value for which the collapse takes place, it is possible to compute an "ϵ-shadow" expansion of canard-values. This characterizes the ϵ-microhalo containing the ϵ-micro-

[6] For comparison the thickness (= duration) of the jump of the solutions in Van der Pol's equation is equal to the ϵ-galaxy [4].





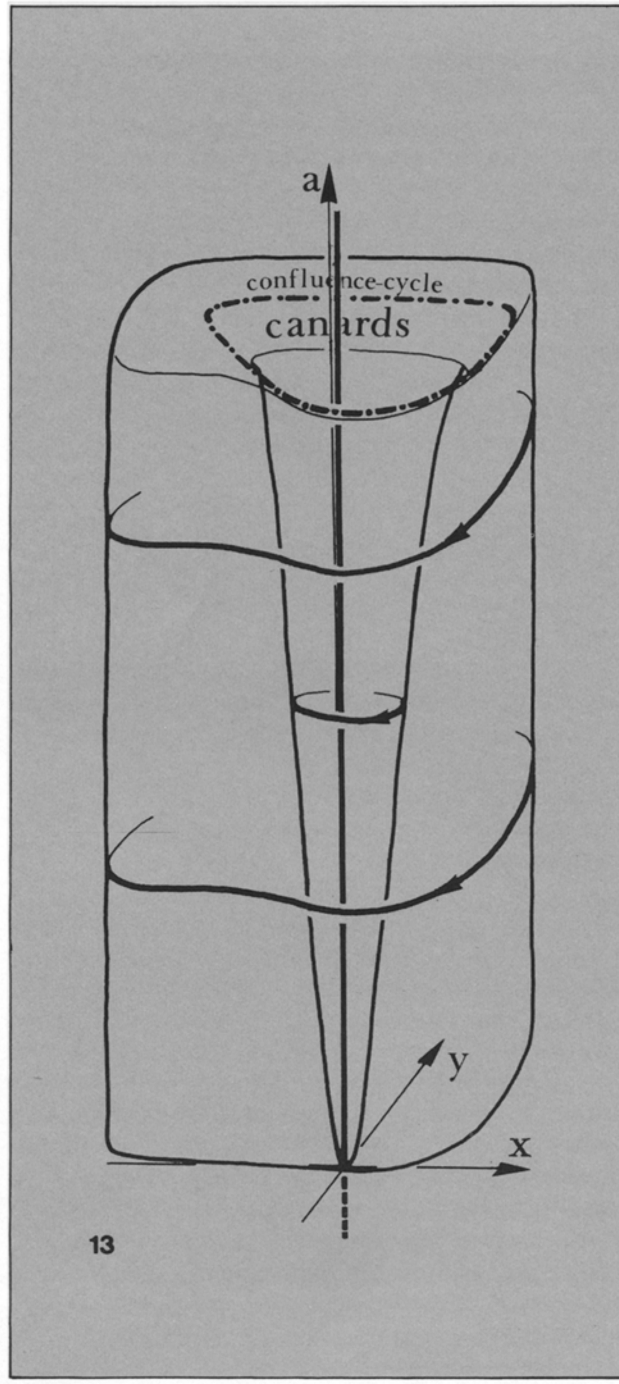
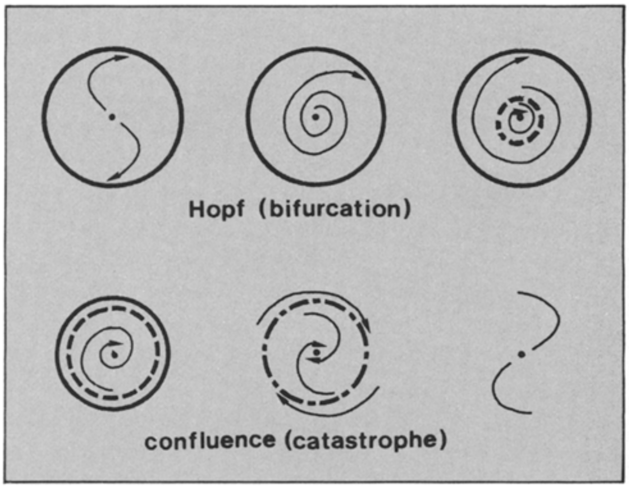


Figure 11. Source (dotted line) and attractor of $\epsilon\ddot{x} + (x^2 - 1)\dot{x} + x = a$, $\epsilon = 0.1$, as a function of a $\in [0.98, 1]$. When a increases towards 1, the size of the cycle diminishes first in a sharp way, the cycle becoming a canard. After that, it diminishes as $\sqrt{1 - a}$ and vanishes in a Hopf bifurcation. Solutions are drawn in the (x, y)-Liénard-plane, with $y = \dot{x} - (x^3/3 - x)$.

Figure 12. (a) Solutions of $\ddot{x} + (\dot{x} + a)^2 + x = 0$ for a = 0.000,001; (for $a = 0$, see figure 3a). All the trajectories of the stippled region leave the river "together" along the lower heavy curve. (b) Comparison between the theoretical results obtained, by rescaling, from the study performed for $\epsilon \simeq 0$, and the actual result, for $\epsilon = 1$. [4]

Figure 13. Sources (dotted line/small cycle) and attractors of $\epsilon\ddot{x} + (5x^4 - 3x^2 + 2x)\dot{x} + x = a$, $\epsilon = 0.1$, as a function of $a \in [0, 0.043]$. For $a = 0$, a small unstable cycle appears in a Hopf bifurcation, and grows as $\sqrt{a}$. After that, the big stable cycle decreases very slowly, and the unstable cycle grows up in a slow, quasi-linear way. Suddenly, for $a \approx 0.042$, stable and unstable cycles, becoming canards jump toward each other and the confluence occurs. For $a > 0.04218189615$, only the stable rest-point (quasi-vertical line) is left. Solutions are drawn in the (x, y)-Liénard-plane, in which $y = \epsilon\dot{x} - (x^5 - x^3 + x^2)$.



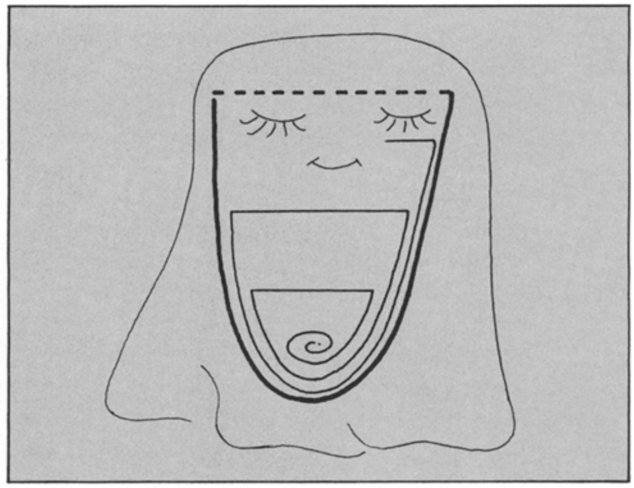


Figure 14. Scheme of classical-(topological) type of the behaviour of the periodic solutions of $\epsilon\ddot{x} + (5x^4 - 3x^2 + 2x)\,\dot{x} + x = a$.

Figure 15. ". . . when the cycle has vanished, the canard is still there, and this ghost via the way-in way-out function, guides the solutions towards a new attractor."

galaxy of the canard-values, for which the drop in size takes place.

The Curtain Effect. Recall that the way-in way-out function applies only along the part of the slow-curve that is followed by at least one canard. In figure 12 are drawn the trajectories of the equation

$$\ddot{x} + (\dot{x} - a)^2 + x = 0,$$

for $a = 10^{-6}$ (and not $a = 0$ as on figure 3a): The canard follows the slow-curve only until the edge of the shaded region, thus reducing the domain of the way-in way-out function to that part of the slow-curve lying outside the shaded and stippled regions. Two things should be pointed out here. Though all trajectories spiral towards the rest-point for all $a > 0$, one notices that there is no discernible change in the behaviour of trajectories outside the shaded or stippled regions where the way-in way-out function still applies. This reflects the fact that this relation does not depend on the value of a (which only determines the *domain* of the function). If the relatively small value of a makes it easy to accept this first observation, it makes the second one all the more striking. All trajectories starting in the stippled region, after having followed the parabola, leave it "together" along the heavy curve bordering the shaded region. This is the neck of the funnel formed by the stippled region; and in a similar way, the heavy curve bordering the stippled region is the neck of the "inverted" funnel through which those trajectories which will fill up the shaded region enter.

This example should make it easier to understand the effect on a solution's behaviour of encountering a bifurcation-value. Changing the value of a is like slowly opening the curtain on a stage, gradually revealing the scene characterized by the way-in way-out function, after which the curtain closes again. During the show, some players, such as cycles, pass from one side of the stage to the other. . . . More prosaically when a bifurcation-with-canards takes place, with long-canard at $a = \bar{a}$, then as a approaches $\bar{a}$, the vector field admits longer and longer canards, and this broadens out gradually the domain of the way-in way-out function. Then, when a crosses $\bar{a}$ and moves away from it, the domain shrinks again. It is this "curtain effect", together with the expression for the way-in way-out function, that provides the analytical description of the changes accompanying a bifurcation.

A Concluding Example. In figure 13 are drawn the source and attractors of the equation

$$\epsilon\ddot{x} + (5x^4 + 3x^2 - 2x)\dot{x} + x = a,\ \epsilon = 0.1.$$

For a increasing near 0, this equation shows successively two bifurcations: a Hopf bifurcation for $a = 0$, from which an unstable cycle arises, and the confluence of a stable and an unstable cycle, for $a \approx 0.042181896$. Classically, the confluence of a source-cycle and an attractor-cycle is perceived as a discontinuity: One limit behaviour vanishes and solutions have to attain their limiting equilibrium state elsewhere. But now it is possible to show that the cycle resulting from the confluence is necessarily a canard: It is at the same time stable and unstable; thus it is not surprising that it is at the same time attracting and repelling. Now, when this cycle has vanished, the canard is still there, and this ghost, via the way-in way-out function, guides the solutions towards a new attractor. As the canards grow and decrease gradually,

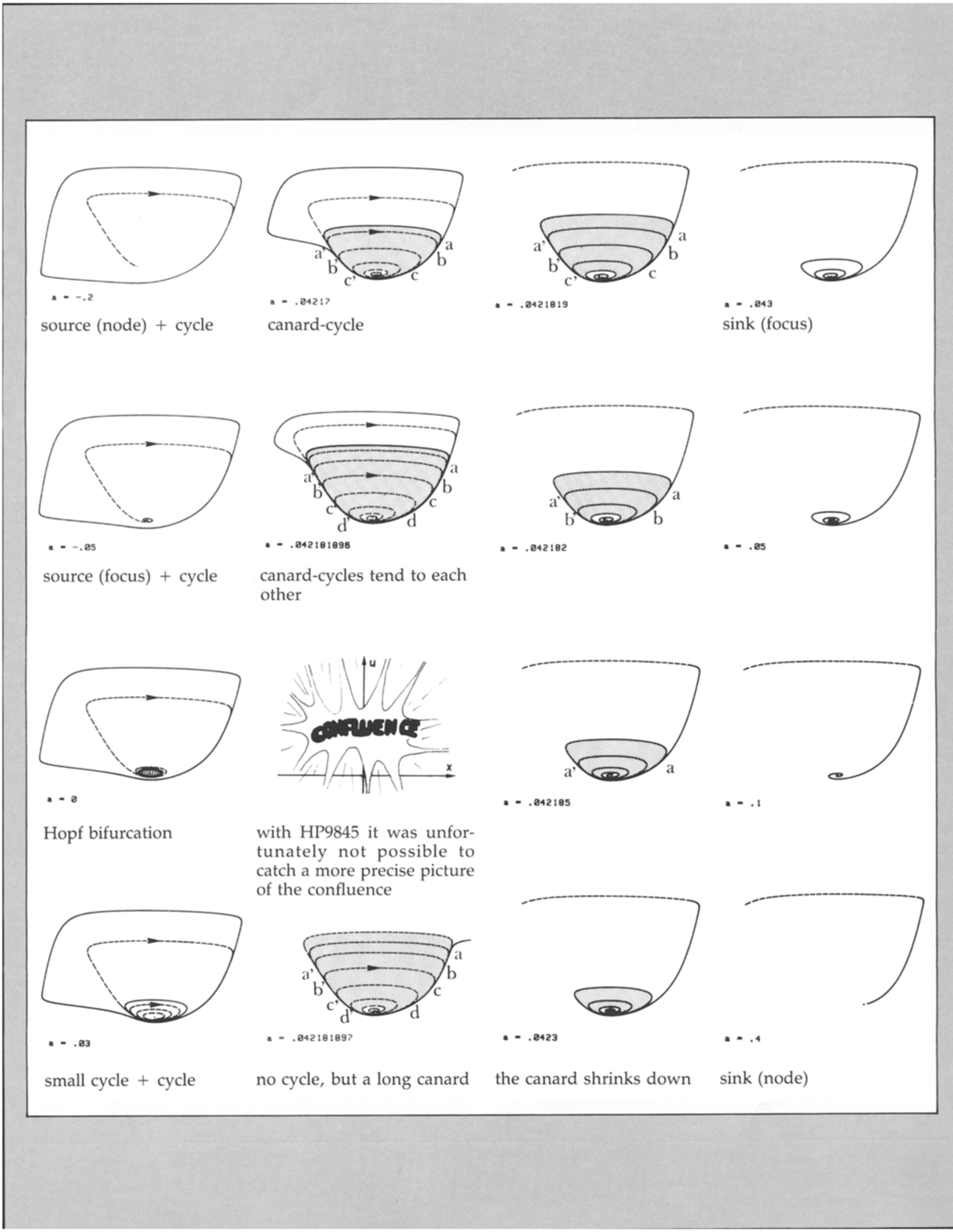


Figure 16. Behavior of the solutions of $\epsilon\ddot{x} + (5x^4 - 3x^2 + 2x)\,\dot{x} + x = a$, for $\epsilon = 0.1$ and a near 0, in the (x, y)-Liénard-plane. First column: The classical Hopf bifurcation. Second and third column: The growing-up and shrinking-down of the canards and of the domain of the way-in way-out function $p \to p'$. Last column: The focus becomes a node.

they allow us to substitute the description of a swift but continuous phenomenon for the previous discontinuous one. Figure 16 shows the complete drama.

Microcomputation and plotting has been performed on an HP 9845 at the "Centre Universitaire de Sidi-Bel-Abbes", and an HP 9825 at the "Centre Universitaire de Tlemcen".

Le texte en français de cet article peut être obtenue auprès de l'auteur, au Département de Mathématiques de l'Université d'Oran, B.P. 1524 Es Senia (Oran), Algérie.

References

1. E. Benoit, C. Lobry (1982) "Les canards de $\mathbf{R}^3$" *C.R. Acad. Sci. Paris,* Série I, *294,* 483–488
2. E. Benoit, J. L. Callot, F. Diener, M. Diener (1981) "Chasse au canard" *Collect. Math. 32,* 37–119
3. I. van den Berg, M. Diener (1981) "Halos et galaxies, une extension du lemme de Robinson" *C.R. Acad. Sc. Paris,* Série I, *293,* 385–388, (1981) "Diverses applications du lemme de Robinson" *C.R. Acad. Sci. Paris,* Série I, *293,* 501–504
4. F. Diener (1980) *Les canards de l'équation* $\ddot{x} + (\dot{x} + a)^2 + x = 0$, Strasbourg: Publications IRMA. (1981) *Méthode du plan d'observabilité,* Strasbourg: Thèse
5. M. Diener (1981) "Etude générique des canards", Strasbourg: Thèse
6. M. Diener, T. Poston "On the perfect delay convention, or the revolt of the slaved variables" in: *Chaos and order in nature* (H. Haken, Ed.) Springer Series in Sinergetics
7. R. Lozi (1982) "Sur un modèle mathématique de suite de bifurcations de motif dans la réaction de Belusov-Zhabotinsky", *C.R. Acad. Sci. Paris,* Série I, *294,* 21–26
8. R. Lutz, M. Goze (1981) *Non standard Analysis. A practical guide with applications,* Springer Lecture Notes in Math., No. 881 Springer-Verlag
9. L. S. Pontryagin (1961) "Asymptotic behaviour of the solutions of the differential equations with small parameter in the high derivatives", *Amer. Math. Soc. Transl.,* Serie 2, *18,* 295–319
10. G. Reeb (1977) "Séance débat sur l'analyse non standard", *Gazette des mathématiciens* No. *8,* 8–14
11. D. Ruelle (février 1980) "Les attracteurs étranges", *La Recherche, 108,* 132–144
12. A. Robinson (1966) *Non Standard Analysis,* Amsterdam: North Holland
13. F. Takens (1976) "Implicit differential equations: some open problems" in: *Singularités d'aplications différentiables, Plan sur Bex 1975,* Springer Lecture Notes in Math., No. *535,* 237–253 Springer-Verlag
14. A. Troesch (1981) *Etude qualitative de systèmes différentiels: une approche fondée sur l'analyse non standard,* Strasbourg: Thèse
15. E. Urlacher (1981) *Oscillations de relaxation et analyse non standard,* Strasbourg: Thèse
16. W. Wasov (1965) *Asymptotic expansions of ordinary differential equations,* New York: Wiley Interscience
17. E. C. Zeeman (1973) "Differential equations for the heartbeat and nerve impulse", in: *Dynamical systems* (M. Peixoto, Ed.) New York: Academic Press, 683–748

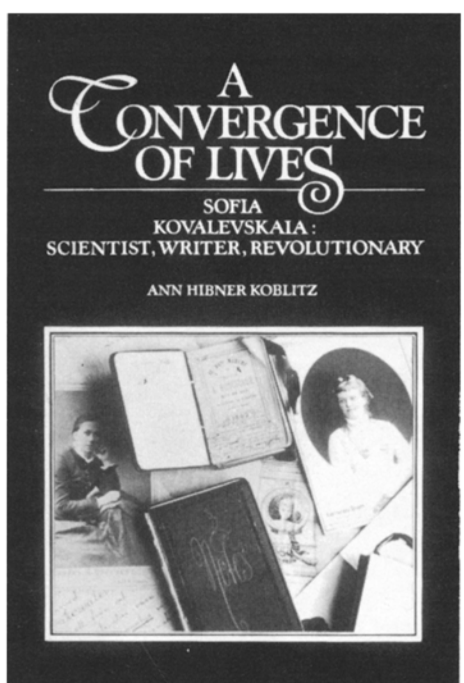
Departement de Mathematiques
Universite d'Oran
B.P. 1524 Es Senia (Oran)
Algerie

A Woman Ahead of Her Time.

"Koblitz' biography... demystif[ies] the woman known in her lifetime as a 'princess of science.' Characterizing Kovalevskaia as 'an extremely gifted but in some ways perfectly ordinary woman,' Koblitz presents a realistic, popular biography [which]... succeeds in integrating Kovalevskaia's science, radical leanings, and literary writings... [and] is especially fine reading for those with a contemporary interest in women in science..."
—Science

To order: Photocopy this ad, and send with check, MasterCard or VISA number (include expiration date and signature) to:

Birkhauser Boston, Inc.
P.O. Box 3005 / Cambridge, MA 02139



A Convergence of Lives
Sofia Kovalevskaia: Scientist, Writer, Revolutionary
by Ann Hibner Koblitz

Mathematician, playwright, feminist, champion of radical political causes, Kovalevskaia was many years ahead of her time. Recent praise for Ann Hibner Koblitz' biography:

"**A Convergence of Lives** is a biography worthy of its subject and respectful of its reader... Koblitz' understanding of Russian and European social, political and intellectual history permeates this enjoyable book, and her analyses and conclusions are particularly enlightening..."
—Physics Today

Dec. 1983 / Hardcover / 304 pages / $19.95 / ISBN 0-8176-3162-3