SIAM J. MATH. ANAL.
Vol. 25, No. 1, pp. 148–173, January 1994

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REGULARIZING MICROSCOPES AND RIVERS*

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This paper is dedicated to Professor Jean-Louis Callot, in memoriam.

Abstract. This paper proposes a generalization of the existing geometric studies of resonance. The Riccati equation associated with any second-order linear equation is extended to any $\mathcal{C}^\infty$ first-order equation. The Morse-critical point is generalized to any "generic" critical point. The resonant solution becomes a general canard solution. The paper explains how to find the regularizing blowup, and shows how classical special functions become enlarged in *rivers*, i.e., some resurgent solutions of polynomial differential equations. The paper shows a matching principle that connects the slow solutions with these rivers. The method to show the existence of canards is applied for some *Union-Jack equations*, i.e., equations with a critical point where three smooth curves intersect.

Key words. resonance, turning points, singular perturbations, matching, canard, river, Union-Jack, macroscope, microscope, nonstandard analysis, Newton polygon

AMS subject classifications. 34E20, 34E05, 03H10

Introduction. For the problem of turning points of singularly perturbed linear second-order differential equations [21], one is indebted to Kopell [14] for a geometric study of *resonances*, remarkable solutions, discovered by Ackerberg and O'Malley [1], of some boundary value problems in the neighbourhood of the equation's turning points.

At the heart of the proof of the main theorem, Kopell introduces a crucial blowup that turns the singular perturbation problem into a regular one. The first approximation of that new equation is a Hermite equation $\ddot{U} - X\dot{U} + kU = 0$ depending on the real parameter k. It is well known that this equation admits, for noninteger k, a basis of solutions, generally denoted by H_k^+ and H_k^-, with polynomial growth at $+\infty$ and $-\infty$, respectively. When k becomes an integer, these two solutions become equal to each other and turn into a polynomial: the Hermite polynomial. As observed independently by Kopell and Callot [5], it is the "crossing" of these two special functions that makes possible (and necessary) the existence of the resonant solutions. We want to show here that the special functions (or more precisely the inverse of their logarithmic derivatives) admit a generalization, the "rivers," that make it possible to solve this kind of turning point problem for a much more general class of equations.

Let's briefly sketch Callot's approach. One first factors out the invariance of the set of solutions of the second-order differential equation $\varepsilon\ddot{u} - f(x)\dot{u} + g(x,a)u = 0$ by considering the new unknown $y = u/\dot{u}$. So, one considers the slow-fast Riccati equation $\varepsilon dy/dx = -f(x)y + g(x,a)y^2 + \varepsilon$ whose slow curve $\{g(x,a)y^2 - f(x)y = 0\}$ is the union of the two smooth curves $\mathcal{C}_1 = \{g(x,a)y = f(x)\}$ and $\mathcal{C}_2 = \{u = 0\}$ (see Fig. 1(a)). In those variables, a resonant solution is a slow solution (i.e., infinitely close to the slow curve) staying infinitely close to $\mathcal{C}_1$ on both sides of some "critical point" x_0, i.e., here, of a point such that $f(x_0) = 0$, and such that, near x_0, $\mathcal{C}_1$ is attracting for $x < x_0$ and repelling for $x > x_0$. This is precisely what is usually called a *canard* [3], [10].

Now comes the trick we want to consider here in a general set-up. In order to determine the behaviour of the slow solutions when they come across the halo*[1] of

* Received by the editors September 16, 1991; accepted for publication (in revised form) January 14, 1993.

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[1] We use here the methods of nonstandard asymptotics. We recall in the appendix (§4) most nonstandard definitions and results involved here: we indicate by a star * the words defined there.

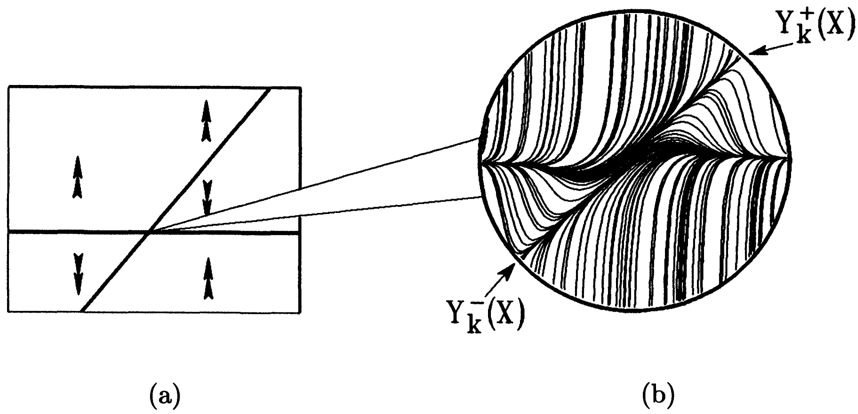

FIG. 1. *Sketch of Callot's proof of the existence of resonant solutions.* (a) *The slow curve of the Riccati equation* $\varepsilon dy/dx = -f(x)y + g(x,a)y^2 + \varepsilon$; (b) *The rivers* $Y^{\pm}(X)$ *of the Riccati–Hermite equation* $dY/dX = -XY + kY^2 + 1$ *for* k *near to* 1. *For larger* k, *the river would have an analogous behaviour at infinity, but would exhibit* $\operatorname{Int} k - 1$ *simple poles.*

the critical point $(x_0, 0)$ (and see how to choose the value of the parameter a in order that such a solution is a canard), one introduces a microscope

$$x - x_0 = \sqrt{\varepsilon} X, \qquad y = \sqrt{\varepsilon} Y$$

that is *regularizing*, in the sense that it turns the singular perturbation of the Riccati equation into a *regular* perturbation of the so-called Riccati–Hermite equation

$$dY/dX = -XY + kY^2 + 1,$$

where $k = k(a) := g(x_0, a)/f'(x_0)$.

Let $Y_k^{\pm} := H_k^{\pm}/\dot{H}_k^{\pm}$. The function Y_k^+ and Y_k^- can clearly be perceived in Fig. 1(b): they are the only trajectories asymptotic to a (neither horizontal nor vertical) straight line, here $X = kY$. The striking fact about these solutions is that the other nearby solutions depart from it in an "exponential way." This kind of behaviour has been studied since that time for general polynomial differential equations, and these solutions are now called *rivers* (see below).

Let $U \subseteq \mathbb{R}^2$ be a nonempty standard open set, $\varepsilon > 0$ an infinitesimal, and f a function defined on U, with regular ε-shadow expansion*.

In this paper, we consider the general nonlinear singular perturbation problem associated with equation

(1) $$\varepsilon dy/dx = f(x,y) \quad (\varepsilon > 0 \text{ infinitesimal}),$$

where $f(x,y) = f_0(x,y) + \varepsilon f_1(x,y) + \cdots$ with (f_n) a standard sequence of $\mathcal{C}^\infty$ functions of $(x,y) \in U$. One may think of $f(x,y) = F(x,y,\varepsilon)$ for some standard smooth function F, but it could as well be $f(x,y) := G(x,y,\varepsilon,\overline{a})$ with $\overline{a}$ the sum to the smallest term of some diverging series $\sum a_n \varepsilon^n$.

The purpose of this paper is first to show how to associate with any equation (1) one, or if necessary several, *regularizing microscopes* in the neighbourhood of a *critical* point of the slow curve $f_0(x,y) = 0$, one for each growth type of the branches of this curve. This makes it possible to convert the singular perturbation into the regular perturbation of one or more *polynomial* differential equation: the *local models.*

We shall then see that the special functions of the above example are not a "miracle": the local model indeed has rivers; the rivers are solutions that are transcendent, but, as in the case of the special functions of physics such as the Airy function, they have remarkable asymptotic behaviour that lead to efficient numerical approximations. We shall recall the definition and main properties of the rivers at §1.2.

The final problem is to relate the behaviour of the slow solutions of (1) when they are infinitely close to the critical point with the rivers of the local model. This study will lead to Theorem 3.1, a matching principle, which is the central result of this paper. It shows how these rivers generalize the special functions $Y_k^{\pm}$ introduced in the recalled study, and make it possible to specify the behaviour of the slow solutions of (1) near the critical points.

In the introduction to §2, we shall give an example in which the critical point is the intersection of three smooth curves. Using a transversality argument (the "crossing" of two rivers) on the local model, we will show (Corollary 3.2) the existence of canards.

1. Slow-curve branches, Newton polygons, and rivers. The local models near critical points of the slow curve depend strongly on the geometry of this slow curve at that point. We give here some elementary tools that yield crucial information about that geometry from a finite number of terms of the Taylor expansion that give dominant balance near the critical point.

This first section will recall some definitions and define notation that we shall need in the sequel concerning the branches of curves $f(x,y) = 0$. Then it will introduce our main tool, the rivers.

1.1. Branches of generic $\mathcal{C}^\infty$ curves.

DEFINITIONS. Let f be a function defined in the neighbourhood of (x_0, y_0). Let φ be a continuous function, whose domain $\mathcal{D}(\varphi)$ is a closed interval with nonempty interior, and having x_0 as one of its ends.We call φ a *branch* at (x_0, y_0) of the curve $\mathcal{C}$ of equation $f(x,y) = f(x_0,y_0)$ if $\varphi(x_0) = y_0$ and $f(x,\varphi(x)) = f(x_0,y_0)$ for all $x \in \mathcal{D}(\varphi)$. One specifies that φ is a *positive* or *negative* branch, according to the sign of $x - x_0$ for $x \in \mathcal{D}(\varphi)$, $x \neq x_0$. Let k and r be real numbers, $k \neq 0$ if $r \neq 0$. We say that φ is a branch of type (k,r) at(x_0,y_0) (or, for short, a (k,r)-branch) if $\varphi(x) - y_0 \sim k|x - x_0|^r$ when $x \to x_0$ in $\mathcal{D}(\varphi)$ in the case $k \neq 0$, or $\varphi(x) \equiv y_0 (= 0)$ in the case $k = r = 0$.

We denote by $\sim$ the classical relation "is asymptotic to" in the neighbourhood of x_0 or $\pm\infty$, according to the context. In nonstandard words this is equivalent, for standard φ, x_0, $k \neq 0$, and r, to $\varphi(x) = y_0 + k|x - x_0|^r(1 + \delta)$ with $\delta \simeq 0$ as soon as $x \simeq x_0$ if $x_0 \in \mathbb{R}$. If $x_0 = \pm\infty$, it means $\varphi(x) = kx^r(1 + \delta)$ with $\delta \simeq 0$ as soon as x is unlimited*, of the sign of $\pm \in \{-1, +1\}$.

Puiseux's theory [12], [15] shows that if f is analytic in the neighbourhood of (x_0, y_0), the curve $\mathcal{C}$ is a union of (k,r)-branches with r rational (and possibly the straight line $x = x_0$). The assumption that f is $\mathcal{C}^\infty$ is much too flabby to force an analogous result, as $\mathcal{C}$ could be any closed subset of $\mathbb{R}^2$. Nevertheless one recovers an analogous result if f is not too degenerate; for that purpose, we introduce now what we shall call the *lower* Newton polygon $\mathcal{N}(f\ ; x_0, y_0)$. Proposition 1.2 will show how to determine the growth types of the various *branches* of $f = 0$ using this lower Newton polygon and the polynomial ${}_rf_0$ that we shall now define (see also Fig. 2).

DEFINITION. Let f be a $\mathcal{C}^\infty$ function on a neighbourhood of $(x_0, y_0) \in \mathbb{R}^2$. We call the *Taylor set* of f at the point (x_0, y_0) the set of couples of integers $\mathcal{T}(f; x_0, y_0) := \{(m,n) \neq (0,0) | f^{(m+n)}_{x^m y^n}(x_0, y_0) \neq 0\}$; if $(x_0, y_0) = (0,0)$ we'll just write $\mathcal{T}(f)$ for

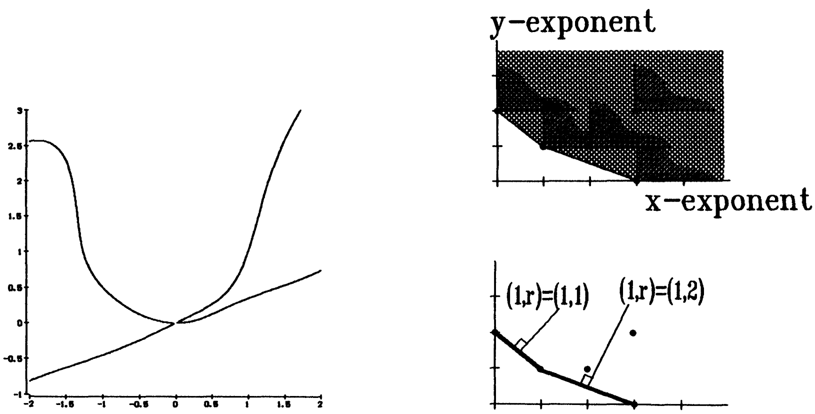

FIG. 2. *The curve $f = 0$ and the lower Newton polygon at $(0,0)$ associated with $f(x,y) = (x-2y)(2y-x^2)+x^3y^2\cos(x-y)$; here ${}_1f(x,y) = 2y(x-2y)$ and ${}_2f(x,y) = x(2y-x^2)$. The curve shows two negative and two positive branches at $(0,0)$, of growth-type $x/2$ and $x^2/2$ at $(0,0)$.*

$\mathcal{T}(f\,;0,0)$. Let $\mathcal{E} = \mathcal{E}(f;x_0,y_0)$ be the convex hull in $\mathbb{R}^2_{mn}$ of the union of all quadrants $m \geq \overline{m}$ and $n \geq \overline{n}$ for all $(\overline{m},\overline{n}) \in \mathcal{T}(f;x_0,y_0)$.

The (*lower*) *Newton polygon* of f at the point (x_0,y_0) is the union $\mathcal{N}(f;x_0,y_0)$ of all oblique segments that build the border of $\mathcal{E}(f;x_0,y_0)$. If $(x_0,y_0) = (0,0)$, we'll just write $\mathcal{N}(f)$ for $\mathcal{N}(f\,;0,0)$.

We call *coslope* of a nonhorizontal segment σ the real number r such that $(1,r)$ is orthogonal to σ. If r is the coslope of some segment σ contained in $\mathcal{N}(f;x_0,y_0)$ we say that r is a coslope of $\mathcal{N}(f;x_0,y_0)$; r is then necessarily a positive rational (see Fig. 2). If there is no point $(m,n) \in \mathcal{N}(f;x_0,y_0)$ such that $n = 0$ (i.e., if $(y-y_0)$ is a factor of $f(x,y) - f(x_0,y_0)$, we shall say that $r = 0$ is (also) a coslope of $\mathcal{N}(f;x_0,y_0)$.

In §1.2 we shall define an *upper* Newton polygon for f any *polynomial.*

DEFINITION. We assume that $\mathcal{T}(f;x_0,y_0) \neq \emptyset$. We define the *r-valuation* of f at the point (x_0,y_0) as the minimum $\mu_r(f;x_0,y_0)$ of all $m+rn$ for $(m,n) \in \mathcal{T}(f;x_0,y_0)$. We'll just write $\mu_r(f)$ when $(x_0,y_0) = (0,0)$. We denote by ${}_rf(X,Y)$ the polynomial, sum of all monomials of the Taylor expansion of f at the point (x_0,y_0) with r-valuation equal to $\mu := \mu_r(f;x_0,y_0)$:

$$ {}_rf(X,Y) := \sum_{m+rn=\mu} \frac{1}{m\,!n\,!} f^{(m+n)}_{x^my^n}(x_0,y_0)X^mY^n. $$

Remark. The polynomial ${}_rf$ is *r-homogeneous* of *r-degree*$=\mu := \mu_r(f;x_0,y_0)$, that is, ${}_rf(\lambda X, \lambda^r Y) = \lambda^{\mu}{}_rf(X,Y)$ for any $\lambda > 0$. For $r \neq 0$, its Taylor set is contained in the segment σ of $\mathcal{N}(f;x_0,y_0)$ with coslope r if any and then $\mathcal{N}({}_rf;x_0,y_0) = \sigma$, or is just equal to one point of $\mathcal{N}(f;x_0,y_0)$.

LEMMA 1.1. *Let $f \in C^\infty$ be a standard function, such that $f(0,0) = 0$ and $\mathcal{T}(f) \neq \emptyset$. Let $\alpha > 0$, $\alpha \simeq 0$, and $r \in \mathbb{R}^+$ standard. Set $\mu = \mu_r(f)$, $x = \alpha X$, and $y = \alpha^r Y$. For all limited (X,Y), one has $f(x,y) = \alpha^\mu({}_rf(X,Y) + \oslash)$.*

Proof. As f and r are standard, so* is μ. Let N be a standard integer, $N \geq \operatorname{Max}\{\mu, \mu/r\}$. We write f as $f = T_Nf + R_Nf$, where T_Nf is the Taylor polynomial of f at $(0,0)$ of degree N, and $R_N f$ is the rest of the Taylor expansion. For all infinitesimal (x,y), one has $R_N f(x,y) = (\mathrm{Max}\{|x|,|y|\})^{N+1}\pounds$ ($\pounds$ denoting a generic limited real number), and thus $R_N f(\alpha X, \alpha^r Y) = \alpha^\mu \oslash$ as $N+1 \geq \mathrm{Max}\{\mu, \mu/r\}$.

Thus, it suffices to show the lemma in the case where f is a standard polynomial $T_N f(x,y)$; this case is trivial. □

PROPOSITION 1.2. *Let $f \in \mathcal{C}^\infty$ be any function defined on some neighbourhood $(0,0)$, such that $f(0,0)=0$ and $\mathcal{T}(f) \neq \emptyset$. Let k and r be nonzero numbers. If the curve $f=0$ admits a negative (k,r)-branch at $(0,0)$, then r is a coslope of $\mathcal{N}(f)$ and ${}_rf(-1,k)=0$.*

Proof. By transfer*, we may assume that f and the branch φ are standard. Since φ is standard, and as $\varphi(x) \sim k|x|^r$, the real numbers k and r are standard. Let $\alpha > 0$ be any infinitesimal. The hypothesis $\varphi(x) \sim k|x|^r$ for $x<0$ implies that $\varphi(-\alpha) = k\alpha^r(1+\oslash)$. Let $\mu := \mu_r(f\ ;0,0)$. Lemma 1.1 implies that

$$f(-\alpha, \varphi(-\alpha)) = f(-\alpha, k\alpha^r(1+\oslash)) = \alpha^\mu({}_rf(-1,k(1+\oslash)) + \delta)$$

with $\delta \simeq 0$. Dividing by α^μ the relation $f(\alpha,\varphi(\alpha))=0$, one gets ${}_rf(-1,k(1+\oslash)) \simeq 0$. Since ${}_rf$ is a standard continuous function, one has

$${}_rf(-1,k) = {}_rf({}^o(-1,k(1+\oslash))) = {}^o({}_rf(-1,k(1+\oslash))) = 0.$$

Thus ${}_rf(-1,k)=0$; since $k \neq 0$, the polynomial ${}_rf(X,Y)$ cannot be just equal to one monomial; r is thus a coslope of $\mathcal{N}(f)$. □

Here is a somewhat more general result that we will *not* use here.

PROPOSITION 1.3. *Let $f \in \mathcal{C}^\infty$ be standard such that $f(0,0)=0$ and $\mathcal{T}(f) \neq \emptyset$. Let $(\alpha,\phi) \simeq (0,0)$, $\alpha > 0$, such that $f(\alpha,\phi)=0$. Assume that $\phi = \alpha^r$ with r appreciable. Then some standard $k_0 \neq 0$ and some $r_0 = {}^o r$ exist such that $\phi = k_0\alpha^{r_0}(1+\oslash)$; r_0 is a coslope of $\mathcal{N}(f;x_0,y_0)$ and ${}_{r_0}f(1,k_0)=0$.*

Proof. Let $r_0 := {}^o r \neq 0$, and $\mu := \mu_{r_0}(f)$, which is standard, as are f and r_0. Let $N \geq \mathrm{Max}\{\mu,\mu/r\}$. Using the factorization $\alpha^r = \alpha^{r-r_0}\alpha^{r_0}$ and reasoning as in proof of Lemma 1.1, one checks that $f(\alpha,\phi) = \alpha^\mu\left({}_{r_0}f(1,\alpha^{r-r_0}) + \oslash\right)$. Since$f(\alpha,\phi)=0$, dividing by α^μ, one sees that ${}_{r_0}f(1,\alpha^{r-r_0}) \simeq 0$. Since $p(K) := {}_{r_0}f(1,K)$ is a nonconstant standard polynomial, thus unlimited for any unlimited K, $k := \alpha^{r-r_0}$ has to be limited, thus near-standard in $\mathbb{R}$. Let $k_0 := {}^o k$. One has $0 = {}^o({}_{r_0}f(1,k)) = {}_{r_0}f(1,{}^o k) = {}_{r_0}f(1,k_0)$. We have to show that $k_0 \neq 0$. If we set $s := 1/r$ and $g(x,y) := f(y,x)$, as $\alpha = \phi^s$, the previous reasoning shows that $l := \alpha^{s-{}^o s}$ is near-standard in $\mathbb{R}$, and $\alpha = l\phi^{s_0}$ with $s_0 := {}^o s$. Thus $\phi = k\alpha^{r_0} = kl^{r_0}\phi^{r_0 s_0} = kl^{r_0}\phi$, with k and l limited. Finally, $1 = kl^{r_0}$ and ${}^o k \neq 0$. □

1.2. Rivers of polynomial differential equations. The polynomial differential equations that occur as local models for the behaviour of the slow solutions near the critical points of the slow curve have a few remarkable solutions called “rivers” that “organize” the qualitative behaviour of the other solutions. These solutions are of polynomial growth and attract or repel exponentially the nearby solutions. These rivers generalize the (logarithmic derivative) of the special functions, the distinguished solutions of (the Riccati equations associated with) the second-order linear differential equations occuring in mathematical physics. We recall here briefly the definition of rivers of a polynomial differential equation and the effective methods to determine the rivers using an *upper* Newton polygon and the polynomials ${}_rP$ associated with P that we also define now. For more details see [11], [7], and [20].

Let us consider the following differential equation:

$$\frac{dY}{dX} = P(X,Y) \tag{2}$$

with $P(X,Y)$ any polynomial with real coefficients.

DEFINITION (see [4]). Let $\overline{Y}$ be any solution of (2), and k and r two real numbers, with $k \neq 0$ if $r \neq 0$. We say that $\overline{Y}$ is a solution of type (k,r) of (2) at $X = +\infty$ (respectively, $X = -\infty$) if there exists a real number X_0 such that $\overline{Y}$ is defined on $[X_0, +\infty)$ (respectively, $(-\infty, X_0]$), and if

$$\overline{Y}(X) \sim k|X|^r \quad \text{at } X = +\infty \quad (\text{respectively, } X = -\infty).$$

We say that $\overline{Y}$ is a *river* of type (k,r) of (2) at $X = +\infty$ (respectively, $X = -\infty$) if $\overline{Y}$ is a solution of type (k,r) of (2) at $X = +\infty$ (respectively, $X = -\infty$) and if

$$\lim_{X\to\pm\infty} X \cdot P'_Y(X, \overline{Y}(X)) = \pm\infty.$$

We shall use the expression "of growth type kX^r at $X = \pm\infty$" as a synonym "of type (k,r) at $X = \pm\infty$."

This last hypothesis suffices to ensure the exponential attracting or repelling of nearby solutions (see [4]). Here come two definitions introducing objects useful to determine the rivers of a polynomial differential equation.

DEFINITION. Let $P := \sum a_{mn}X^mY^n$, and let $\mathcal{D} = \mathcal{D}(P)$ be the convex hull in $\mathbb{R}^2_+$ of the half-lines $m \leq \overline{m}$ and $n = \overline{n}$, for all $(\overline{m}, \overline{n})$ such that $a_{\overline{m}\overline{n}} \neq 0$. We call the *upper* Newton polygon of the polynomial P the union $\mathcal{M}(P)$ of all oblique segments of the border of $\mathcal{D}$.

If σ is a segment contained in $\mathcal{M}$ of coslope r, we say that r is a coslope of $\mathcal{M}(P)$. If $(Y-k)$ is a factor of P for some $k \in \mathbb{R}$, we shall say that $r = 0$ is (also) a coslope of $\mathcal{M}(P)$.

DEFINITION. Let $r \in \mathbb{R}$ and $P(X,Y) := \sum a_{mn}X^mY^n$ for any polynomial. We define the r-degree of P to be the number $\partial_r P := \operatorname{Max}\{m + rn \mid a_{mn} \neq 0\}$. We set

$$^rP(X,Y) = \sum_{m+rn=\partial_r P} a_{mn}X^mY^n.$$

Remarks. As we already pointed out, for any smooth function f at (x_0, y_0), the polynomial $_rf$ is *r-homogeneous* of *r-degree*=$\mu := \mu_r(f; x_0, y_0)$, and, if $r \neq 0$, its Taylor set is contained in the segment σ of $\mathcal{N}(f; x_0, y_0)$ with coslope r if any and then $\mathcal{N}(_rf; x_0, y_0) = \sigma$, or is just equal to one point. Thus $\mathcal{N}(_rf; x_0, y_0) = \mathcal{M}(_rf)$, $^r(_rf) = {_rf} = {_r(_rf)}$ and $\partial_r(_rf) = \mu_r(f\,; x_0, y_0)$.

By construction of $\mathcal{N}(_rf; x_0, y_0)$, any of its coslopes r are nonnegative; this was done intentionally, for the sake of simplicity, choosing to define $\mathcal{E}$ as the convex hull of a union of quadrants. Indeed, a curve of type (k,r) at (x_0, y_0) with $r < 0$ would tend to infinity when x goes to x_0; the problem would no longer be local so we would need some rigidity on f, such as "f is a polynomial in y^α" (this is beyond the purpose of this paper); nevertheless, in §3 we give an example on singular deformation to show how curves of type (k,r) with $r < 0$ arise.

As we consider here a river solution of *polynomial* differential equations, it is no longer necessary to exclude the case $r < 0$: this is why the upper Newton polygon was introduced as the convex hull of horizontal half-lines instead of quadrants as in the previous case. So $\mathcal{M}(P)$ may have some negative r as coslope.

There is a result [11], [4] in which (a) and (b) are the analogs, for rivers, of Proposition 1.2 for the branches of the slow curve at a critical point; condition (c) is related to the behaviour of the other solutions with respect to the river: for X unlimited (that is, for $x := \varepsilon X \not\simeq 0$, where $\varepsilon \simeq 0$), the river behaves as a slow solution of a (usually other) slow-fast differential equation. As rivers are concerned with the behaviour of the equation at infinity, it is the *upper* Newton polygon that is of interest here. Proposition 2.1, Theorem 2.2, and finally the matching principle 3.1 will make precise the relation between growth type of a branch at a critical point and rivers of the related local model under the regularizing blowup.

PROPOSITION 1.4. *If* (2) *has a solution of type* (k, r) *at* $X = \pm\infty$, *then* (a) r *is a coslope of the upper Newton polygon* $\mathcal{M}(P)$; (b) ${}^{r}P(\pm 1, k) = 0$. *Moreover, this solution is a river at* $X = \pm\infty$ *if and only if* (c) $c(r) := 1 - r + \partial_r P > 0$.

Conversely, if beyond properties (a), (b), and (c) above, one has

$$ {}^{r}P'_Y(1, k) \neq 0, $$

then the *rivers-existence theorem* [11] implies that (2) has indeed a river of type (k, r).

One of the interests of the river solutions lies in the fact that they admit, as most special functions, a (diverging) asymptotic expansion of Gevrey type [7], and that they are resurgent [13], [6] and, in particular, summable in the sense of the summation of diverging series [17]. In practice, in the "good cases," i.e., if ${}^{r}P'_Y(\pm 1, k) \neq 0$ and for $r = p/q$ with p and $q > 0$ integers, any river has an asymptotic expansion $kx^r \sum_{n \geq 0} a_n x^{-n/q}$ for which it is easy to compute as many terms as desirable, with help, for example, from programs such as *Maple* or *Mathematica.* The fact that the expansion is of Gevrey type, that is, (a_n) grows no faster as $n\,!$, implies among other things [19] that the error committed when truncating the sum is of order equal to the first "neglected" term. So one may compute approximations of the rivers of order of the smallest term $a_n x^{-n/q}$ ("summation up to the smallest term"), or excellent approximation with few terms, as for the diverging expansions of special functions. The fact that the expansion is resurgent is related to its "transasymptotic" expansion (exponential corrections). So rivers lend themselves nicely to numerical computations.

2. Regularizing microscopes and local models.

Examples and definition. Let us consider the following example of slow-fast differential equation (Fig. 3):

$$ \varepsilon dy/dx = (y - ax)(y - bx)(y - cx) + p(x, y), \quad \text{with } abc \neq 0 \tag{3} $$

with $a < b < c$ three standard real numbers, and p a standard $\mathcal{C}^\infty$ function with $\mathcal{T}_3(p) \equiv 0$, i.e., zero Taylor polynomial of degree 3. The slow curve is the union of three smooth curves passing through the origin, and tangent to the three straight lines $y = ax$, $y = bx$, and $y = cx$, respectively, that motivate the name, *Union-Jack* equation, that we give to it; $(0, 0)$ is a more degenerate critical point than in the case of a Riccati equation. Let us perform the following change of variable:

$$ \alpha X = x, \quad \alpha Y = y, \quad \text{with } \alpha = \varepsilon^{1/3}. \tag{4} $$

Since $\varepsilon \simeq 0$, this operates as a microscope on the phase-space, and the image is a blowup of this phase space. The behaviour of the solutions for (X, Y) limited corresponds to the behaviour of solutions of (3) in the α-galaxy of $(0, 0)$. Equation (3) becomes

$$ dY/dX = (Y - aX)(Y - bX)(Y - cX) + \varepsilon^{-1} p(\alpha X, \alpha Y). \tag{5} $$

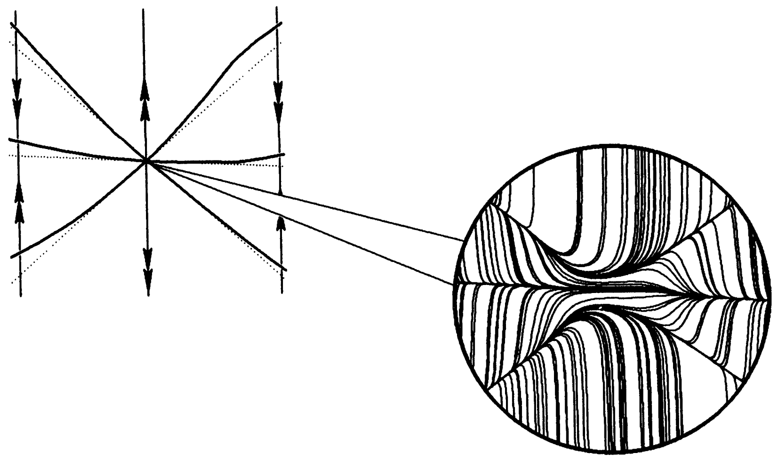

FIG. 3. *Slow-curve of a Union-Jack equation* 3 *and its image under a regularizing blowup.*

Since, by assumption, the Taylor polynomial of degree 3 of p is zero, we see that the term $\varepsilon^{-1}p(\alpha X, \alpha Y)$ is infinitesimal for (X, Y) limited. In other words, the blow up (4) made it possible, in the vicinity (i.e., infinitesimal) of the critical point, to convert the singular perturbed equation (3) into a regular one. The shadow of the solutions of (5) are solutions of the standard differential equation

$$dY/dX = (Y - aX)(Y - bX)(Y - cX). \tag{6}$$

And this is now a polynomial differential equation. An easy study of its rivers shows that this equation has three rivers at $X = +\infty$, and as many at $X = -\infty$, all being asymptotic to kX^r for some $k = a, b, c$, and $r = 1$. Theorem 3.1 will show that any slow solution following $\{y = bx\}$ for $x < 0$ (attracting solution), or $\{y = ax\}$ or $\{y = cx\}$ for $x > 0$ (repelling solution), is infinitely close to the corresponding river at the scale of (X, Y).

In this study, we want to elucidate a double "miracle": how to associate with any critical point of a large class such a desingularizing blowup, and how to approximate the slow solutions in the very vicinity of the critical point by a river of some standard, polynomial, differential equation. As an example of possible application of that method, we shall show how to deduce the existence of *canards* in a one-parameter family of Union-Jack equations.

DEFINITION. Let $\varphi(x) \sim y_0 + k(x - x_0)^r$ be a positive (respectively, negative) branch of the slow curve $\{f_0 = 0\}$ of (1) in the neighbourhood of (x_0, y_0). We call this the *regularizing microscope* of (1) in the neighbourhood of (x_0, y_0) with respect to the branch φ any change of variable $x = x_0 + \alpha X$, $y = y_0 + \beta Y$, with $\alpha > 0$, $\alpha \simeq 0$ and any $\beta > 0$, that converts that equation into $dY/dX = F(X, Y)$, with F of class S^0, ${}^oF \neq 0$, such that the standard equation $dY/dX = {}^oF(X, Y)$ admits a solution asymptotic to kX^r at $X = +\infty$ (respectively, $X = -\infty$). Equation $dY/dX = F(X, Y)$ will be called a *regularizing blowup* of (1) (by the considered regularizing microscope).

Remark. For small enough $\alpha = \beta$ it is always possible to get an infinitesimal F, that is, a regular perturbation of the trivial equation $dY/dX = 0$. This way to "regularize" the singular perturbation is of course of low interest: it is just a way to express the classical theorem of local straightening of locally Lipschitz differential equations' solutions. This is why we ask for ${}^oF \neq 0$; as we shall see, this needs a more subtle choice of α and β, neither too large nor too small.

2.1. Case $f = f_0$: the simple equation. We first consider the case of the *simple* equation

$$\varepsilon dy/dx = f_0(x,y) \quad (\varepsilon > 0 \text{ infinitesimal}), \tag{7}$$

i.e., the case where f is a standard function. It is easy to see that the microscope

$$x = x_0 + \alpha X, \qquad y = y_0 + \alpha^r Y, \tag{8}$$

centered at a standard point (x_0, y_0) of the slow curve, yields the following blowup of (7):

$$dY/dX = (\alpha^{1-r}/\varepsilon) f(x_0 + \alpha X, y_0 + \alpha^r Y) = (\alpha^{1-r+\mu}/\varepsilon)\left({}_r f_0(X,Y) + \delta\right),$$

with $\mu = \mu_r(f_0\,; x_0, y_0)$ and $\delta \simeq 0$ for any limited (X,Y). In order to get a regularizing microscope, one has to consider those choices of α such that $\alpha^{1-r+\mu}/\varepsilon$ is appreciable, for example, equal to 1, which means that $\alpha = \varepsilon^{1/(1-r+\mu)}$. The crucial point of this study consists in observing that for some convenient choices of r the shadow of the near-standard equation associated in that way is a *polynomial equation exhibiting rivers.*

PROPOSITION 2.1 (case of the simple equation). *Assume f is standard (i.e., $f = f_0$) and that $(0,0)$ belongs to the slow curve $\{f_0 = 0\}$. Let $r \geq 0$ be any coslope of the lower Newton polygon $\mathcal{N}(f_0)$, $\mu = \mu_r(f_0)$, and k be any root of (the algebraic) equation ${}_r(f_0)(\pm 1, k) = 0$. Let $s = 1/(1 - r + \mu)$. In that case $s > 0$, $\alpha := \varepsilon^s$ is infinitesimal, the microscope*

$$x = \varepsilon^s X, \qquad y = \varepsilon^{sr} Y \tag{9}$$

is regularizing for (7), and the resulting blowup is infinitely close to the polynomial equation

$$dY/dX = {}_r f_0(X,Y). \tag{10}$$

If $({}_r f_0)'_Y(\pm 1, k) \neq 0$, this equation has a river of type kX^r at $X = \pm\infty$.

Proof. One has $1/(1-r+\mu) > 0$: let $(m,n) \in \mathcal{N}(f_0)$ be the point of the segment σ of the lower Newton polygon of f_0 of coslope r that has the largest ordinate n; one has $n \geq 1$ and thus

$$1 - r + \mu = 1 - r + \mu_r(f_0) = 1 - r + m + rn = (1+m) + r(n-1) \geq 1.$$

Thus $1 - r + \mu$ is positive and is standard, whence $\alpha := \varepsilon^{1/(1-r+\mu)} \simeq 0$.

The microscope is regularizing: Lemma 1.1 implies that

$$f(x,y) = \alpha^\mu({}_r f(X,Y) + \delta),$$

with $\delta \simeq 0$ for all limited (X,Y), whence

$$dY/dX = \alpha^{1-r}\alpha^\mu\varepsilon^{-1}\left({}_r f_0(X,Y) + \delta\right) = {}_r f_0(X,Y) + \delta =: F(X,Y),$$

and ${}^oF = {}_r f_0$. The *short-shadow lemma** [9] implies that the shadows of the solutions at this scale really are solutions of (10).

Equation $dY/dX = {}^oF(X,Y)$ does have rivers: the polynomial ${}_rf_0$ is r-homogeneous (of r-degree $\mu := \mu_r(f_0)$), and its *upper* Newton polygon $\mathcal{M}\,({}_rf_0)$ is just equal to the segment σ. Since $({}_rf_0)'_Y\,(\pm 1, k) \neq 0$, the *rivers-existence theorem* [11] implies the existence of one river of type kX^r at $X = \pm\infty$. □

Examples. Case of a regular point of the slow curve. Assume $f = f_0$, i.e., f is standard, and that $(0,0)$ is a regular point of the slow curve of (1), that is, $f_0(0,0) = 0$ and $f_0{}'_x f_0{}'_y(0,0) \neq 0$. The implicit-function theorem implies that the slow curve $\{f_0 = 0\}$ is, in the neighbourhood of $(0,0)$, a standard smooth curve tangent to the straight line $\{y = kx\}$, with $k = -A/B$, where $A := f_0{}'_x(0,0)$ and $B := f_0{}'_y(0,0)$. So, it has two branches (one for $x \geq 0$ and one for $x \leq 0$), of type $(k, 1)$.

The microscope $x = \varepsilon X$, $y = \varepsilon Y$ is regularizing and turns (1) into a near-standard equation, the shadows of the trajectories of which being solutions of

$$\frac{dY}{dX} = AX + BY.$$

This equation has the explicit solution $\overline{Y}(X) := -XA/B - A/B^2$, which is a river, both at $X = +\infty$ and at $X = -\infty$, of the same type as the two branches of the slow curve.

Case of a fold point. In [16], Mishchenko and Rosov study the behaviour of the solutions of (1) near a fold point at $(0,0)$, that is, in the case $f_0{}'_x(0,0) \neq 0$ and $f_0{}'_y(0,0) = 0$, but $f_0{}''_{y^2}(0,0) \neq 0$: the slow curve looks like a horizontal parabola: it has two branches (assumed to be negative) of type $(k, \frac{1}{2})$.

Let $s_0 = \frac{2}{3}$; the microscope $x = \varepsilon^{s_0}X$, $y = \varepsilon^{s_0/2}Y$ is regularizing at such a fold-point. It turns (1) into a near-standard equation, the shadows of the trajectories being solutions of

$$\frac{dY}{dX} = AX + BY^2$$

with $A = f_0{}'_x(0,0)$ and $B = \frac{1}{2}f_0{}''_{y^2}(0,0)$. It is a Liouville equation that exhibits [11] two families of rivers (Fig. 4), one containing just one isolated river (asymptotic to $y = +\sqrt{-x}$ on the figure, and that the authors call the "dividing solution," which they are able, in that case, to express in terms of Bessel functions), and the other containing an infinity of rivers, all asymptotic to each other (and to $-\sqrt{-x}$ on the figure). The types of the two families of rivers are equal to those of the two branches of slow curve.

2.2. Regular deformations. We now come back to the general case, where $f = f_0 + \varepsilon f_1 + \cdots$ that we shall consider in different ways, according to whether the microscope (9) used to regularize the simple (or *simplified*) equation (7), obtained by replacing f by f_0, is still regularizing for the *complete* equation (1) or not.

Let us first consider the example of the following equation:

$$\varepsilon dy/dx = \pm(y-x)(y-x^2) + \varepsilon(a+bx), \quad \text{with } a, b \in \mathbb{R}. \tag{11}$$

The slow curve, with equation $(y-x)(y-x^2) = 0$, exhibits some branches at $(0,0)$ of type x^r with $r = 1$ and $r = 2$. Let's consider successively the cases $r = 1$ and $r = 2$.

For $r = 1$, considering the simplified equation leads to selecting $s = \frac{1}{2}$, the microscope $x = \varepsilon^{1/2}X$, $y = \varepsilon^{1/2}Y$ yields the blowup

$$dY/dX = \pm(Y-X)(Y-\varepsilon^{1/2}X^2) + a + \varepsilon^{1/2}bX, \tag{12}$$

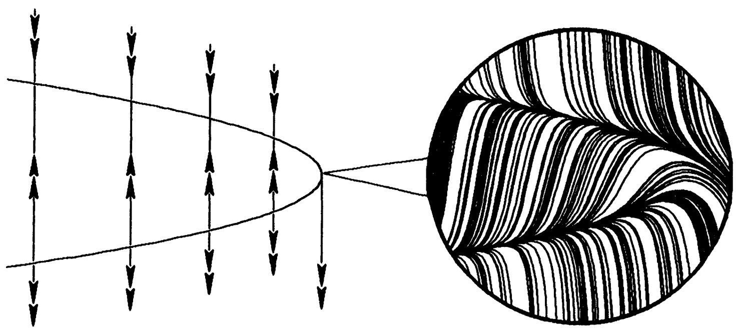

FIG. 4. *Slow-fast equation with slow curve exhibiting a fold point, and its image under a regularizing microscope. This last equation has a unique river asymptotic to $\sqrt{-x}$, and an infinity of rivers all asymptotic to $-\sqrt{-x}$, and actually with the same asymptotic expansion: they are exponentially close to each other.*

which, for limited a and b, is infinitely close to a standard equation (i.e., is a regular perturbation), namely, $dY/dX = \pm Y(Y-X) + {}^\circ a$.

For $r = 2$, considering the simplified equation suggests once more letting $s = \frac{1}{2}$; the microscope $x = \varepsilon^{1/2}X$, $y = \varepsilon Y$ transforms (11) into

$$dY/dX = \mp X(Y - X^2) \pm \varepsilon^{1/2} Y(Y - X^2) + a\varepsilon^{-1/2} + bX. \tag{13}$$

We notice here that if a is standard nonzero, the term $a\varepsilon^{-1/2}$ is unlimited: the microscope that is regularizing for the simplified equation associated with (11) is no longer regularizing for the complete equation (11).

In terms of the following definition, the "deformation term" $\varepsilon(a+bx)$ is a *regular deformation* for the growth-type x^1, and a *singular deformation* for the growth-type x^2 (if ${}^\circ a \neq 0$).

DEFINITION. We say that (1) is a *regular deformation at the point* (x_0, y_0) *of the simple equation* (7) *for the growth type* x^r, if, for $s_0 = 1/(1 - r + \mu_r(f_0; x_0, y_0))$, the microscope

$$x = x_0 + \varepsilon^{s_0} X, \qquad y = y_0 + \varepsilon^{s_0 r} Y \tag{14}$$

is regularizing for (1). If not, we call it a *singular deformation at the point* (x_0, y_0) *of* (7) *for the growth type* x^r.

The purpose of the following theorem is first to show that only an explicit finite number of terms of the expansion of $f = \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$ may introduce a singular deformation of the simple equation for a given branch-growth type. Then it establishes a relation existing between the local model of the simple equation and that of the complete equation, in case of regular deformation. This will make it possible to show (Corollary 2.3) that these two models have essentially the "same kind" of rivers.

THEOREM 2.2. *Let* $s_0 = 1/(1 - r + \mu_r(f_0))$. *A necessary and sufficient condition for* (1) *to be a regular deformation at the point* $(0,0)$ *for the simplified equation for the growth type* x^r *is that, for all* p *such that* $1 \leq p < 1 + s_0(r-1)$, *the following inequality holds:*

$$p - 1 + s_0(1 - r + \mu_r(f_p)) \geq 0. \tag{15}$$

If so, the blowup $dY/dX = F(X,Y)$ *of equation* (1) *by the microscope* (14) *is infinitely close to some standard polynomial differential equation*

$$dY/dX = P(X,Y), \quad \text{with } {}^rP(X,Y) = {}_rf_0,$$

called the local model *of* (1) *at the point* $(x_0, y_0) = (0,0)$ *for branches of growth type* $|x - x_0|^r$.

Proof. Let $p^* > 1 + s_0(r-1)$ be some fixed standard integer and denote $\mu(f_p)$ just by μ_p. One has

$$f(x,y) = f_0(x,y) + \varepsilon f_1(x,y) + \cdots + \varepsilon^{p^*}\left(f_{p^*}(x,y) + \phi\right),$$

for all $(x,y) \simeq (0,0)$; thus, for all limited (X,Y)

$$\begin{aligned} F(X,Y) &= \varepsilon^{-1+s_0(1-r)} f(\varepsilon^{s_0}X, \varepsilon^{rs_0}X) \\ &= \sum_{p<p^*} \varepsilon^{-1+s_0(1-r)+p+s_0\mu_p}\left({}_r f_p(X,Y) + \phi\right) \\ &\quad + \varepsilon^{p^*-1+s_0(1-r)}\left(f_{p^*}(0,0) + \phi\right) \\ &\simeq \sum_{p<p^*} \varepsilon^{-1+s_0(1-r)+p+s_0\mu_p}\left({}_r f_p(X,Y) + \phi\right). \end{aligned}$$

So, the deformation is regular if and only if all the exponents of this later sum are positive or zero, that is, if (15) holds for all $p < 1 + s_0(r-1)$. If so, let $\Pi_* := \{p < p^* \mid p + s_0\mu_p = 1 + s_0(r-1)\}$; so, for any limited (X,Y),

$$F(X,Y) \simeq \sum_{p\in\Pi_*} {}_r f_p(X,Y) =: P(X,Y).$$

Since this defines a standard polynomial P, one finally has ${}^oF = P$.

Still in that case, let $p \geq 1$, $p \leq 1 + s_0(r-1)$, and assume that f_p brings a noninfinitesimal contribution to $F(X,Y)$, that is, assume $-1 + s_0(1-r) + p + s_0\mu_r(f_p) = 0$. By definition of s_0, we thus have $p + s_0\mu_r(f_p) = 0 + s_0\mu_r(f_0)$, whence $\mu_r(f_0) - \mu_r(f_p) = p/s_0$, which is strictly positive, since $p \geq 1$ and $s_0 > 0$ as we saw in Proposition 2.1. Thus $\mu_r(f_0) > \mu_r(f_p)$, and ${}^rP = {}^r({}^oF) = {}_r f_0$. □

Example. If $r \leq 1$, the condition $1 \leq p < 1 + s_0(r-1)$ is never satisfied, and thus any deformation is automatically satisfied for the growth type x^r: at a regular point or at a fold point, or at a Morse point ($f(0,0) = 0$, $\mathrm{Jac}\,(f)(0,0) = (0,0)$, but $\mathrm{hess}\,(f)(0,0) \neq 0$) any deformation is regular for all growth types of the branches that reach such a point. This explains why all the existing studies of canards never came across the problem of singular deformations.

COROLLARY 2.3 (regular deformations). *Assume* $(x_0, y_0) = (0,0)$. *Let* $r \geq 0$ *be a coslope of the lower Newton polygon* $\mathcal{N}(f_0)$, *and* k_0 *any root of the algebraic equation* ${}_r f_0(\pm 1, k)$. *Let* $s_0 = 1/\left(1 - r + \mu_r(f_0)\right)$. *If* (1) *is, at the point* $(0,0)$, *a regular deformation of the simplified equation* (7) *for the growth type* x^r, *the microscope* (14) $x = \varepsilon^{s_0}X$, $y = \varepsilon^{rs_0}Y$ *is regularizing for* (1), *and the image of equation* (1) *by this microscope* (14) *is infinitely close to the standard polynomial equation*

$$dY/dX = P(X,Y), \tag{16}$$

with ${}^rP(X,Y) = {}_r f_0$. *If* $({}_r f_0)'_Y(\pm 1, k_0) \neq 0$, *this equation has a river of type* k_0X^r *at* $X = \pm\infty$.

Proof. The previous theorem implies that ${}^rP = {}_r f_0$, and the existence of rivers of type (k,r) for $dY/dX = P(X,Y)$ depends only on that polynomial ${}^rP$: so the corollary follows immediately from Theorem 2.2. □

We shall come back to the problem of singular deformations at the end of the study of preresonant solutions.

3. Preresonant trajectories.

3.1. Entrance in the halo of a critical point. The two previous sections were dedicated to the geometric study of (1), that is, in some sense, to the formal solutions of that equation. We can now come to the study of the behaviour of the (slow) *solutions* of that equation, when x becomes infinitely close to a critical point. Actually, to know that the solution is slow for x not infinitely close to the critical point does not always give strong information about its behaviour when x becomes infinitely close to that point. Indeed, consider, for example, the equation $\varepsilon dy/dx = -2xy$, which has a critical point at $(0,0)$, and whose solutions are given, as a function of the initial condition $y_- = \overline{y}(x_-)$, by

$$\overline{y}(x) = y_- e^{x_-^2/\varepsilon} e^{-x^2/\varepsilon}.$$

The hypothesis that the solution is equal to an infinitesimal $y_- = \overline{y}(x_-)$ at some "initial condition" $x_- \not\simeq 0$ does not suffice to give some control on $\overline{y}$ for $x \simeq 0$: for example, for $x_- = -1$ and $x = 0$, y_- may be infinitesimal and $\overline{y}(0) = y_- e^{1/\varepsilon}$ may take, according to the value of y_-, any infinitesimal value, and $\overline{y}(0)$ may even be appreciable or illimited, for some convenient choice of $y_- \simeq 0$. This comes, essentially, from the fact that the slow curve $y = 0$ is repelling for $x < 0$ (or attracting for $x > 0$). In the general case, we can get a good precision on the behaviour of slow solutions in the halo of the critical point (x_0, y_0) and more precisely under a regularizing microscope, only for solutions that follow an attracting curve for $x < x_0$, or a repelling one for $x > x_0$. Such a solution, defined and satisfying that condition on *both* sides of x_0 is a *canard* [10], also called [18] "resonant in the sense of N. Kopell." This is why we shall call any solution satisfying that condition on *one* side (at least) of the critical point *pre*resonant.

DEFINITION. Let $\varphi : \mathcal{D}(\varphi) \longrightarrow \mathbb{R}$ be a (k,r)-branch at the point (x_0, y_0) of the slow-curve $f_0(x,y) = 0$ of (1). We say that φ is a *preresonant branch* at (x_0, y_0) if and only if φ is standard, $(f_0)'_y(x_0, y_0) = 0$, and $(f_0)'_y(x, \varphi(x))$ is nonzero and of the same sign as $(x - x_0)$ for all $x \in \mathcal{D}(\varphi)$ not equal to x_0 (i.e., if and only if (x_0, y_0) is critical, and for $x \neq x_0$, $(x, \varphi(x))$ is attracting if $x < x_0$, or repelling if $x > x_0$).

Let φ be a (k,r)-branch at (x_0, y_0) of the slow curve of (1) and $\overline{y}$ a solution of (1). We say that $\overline{y}$ is a *preresonant solution at* (x_0, y_0) *attached to* φ if and only if φ is preresonant, and for all $x \not\simeq x_0$ in $\mathcal{D}(\varphi)$, $\overline{y}(x)$ is defined and $\overline{y}(x) \simeq \varphi(x)$.

Examples. For a Union-Jack equation (3) (with $a < b < c$) (Fig. 3), the slow-curve at $(0,0)$ has one preresonant branch defined for $x < 0$ which is tangent to $y = bx$, and two preresonant branches defined for $x > 0$, tangent to $y = ax$ and $y = cx$, respectively.

At a fold point (Fig. 4), one of the branches of the slow curve is preresonant and the other is not.

To each preresonant branch there corresponds, on the regularizing blowup, an isolated river that is a shadow of the image by the microscope of any preresonant solution infinitely close to that preresonant branch. Theorem 3.1 shows that this is a general fact.

We shall consider in this main theorem the case of a preresonant branch defined for $x < x_0$, with $x_0 = 0$; one has of course an analogous result for any standard x_0, and also for any preresonant branch defined for $x > x_0$.

THEOREM 3.1 (matching principle). *Let* $k \neq 0$, $r > 0$, $x_- < 0$ *be standard, and* $\varphi : [x_-, 0] \longrightarrow \mathbb{R}$ *be a preresonant* (k,r)*-branch at* $(0,0)$ *of the slow curve of* (1); *Let*

$s_0 := 1/(1 - r + \mu_r(f_0))$. *Assume that* $({}_r f_0)'_y(-1, k) \neq 0$, *and that* (1) *is a regular deformation for the growth type of* φ *of the simplified equation*

$$\varepsilon \frac{dy}{dx} = f_0(x, y).$$

Let $P(X, Y)$ *be the polynomial such that the shadow of the blowup of* (1) *by the regularizing microscope*

$$x = \varepsilon^{s_0} X, \qquad y = \varepsilon^{r s_0} Y \tag{17}$$

is

$$\frac{dY}{dX} = P(X, Y). \tag{18}$$

Let $\overline{y}$ *be any maximal solution of* (1) *such that* $\overline{y}(x_-) \simeq \varphi(x_-)$. *Then*

(1) *There exists some limited* $X_+ \leq 0$ *such that* $\overline{y}$ *is defined and preresonant, attached to* φ *on* $[x_-, x_+]$, *with* $x_+ := \varepsilon^{s_0} X_+ \simeq 0$.

(2) *For all* $x \in [x_-, x_+]$ *such that* $x \simeq 0$, *if* x/ε^{s_0} *is unlimited, then* $\overline{y}(x) = k|x|^r(1 + \not{o})$.

(3) *Equation* (18) *has a unique river* $\hat{Y} : (-\infty, a) \longrightarrow \mathbb{R}$ *of type* (k, r) *at* $X = -\infty$; *for any* X *near-standard in* $(-\infty, a)$ *the image* $\overline{Y}(X)$ *of* $\overline{y}(x)$ *by the microscope* (17) *is infinitely close to* $\hat{Y}(X)$. *So for* $x := \varepsilon^{s_0} X$, $\overline{y}(x)$ *is defined and satisfies*

$$\varepsilon^{-r s_0} \overline{y}(\varepsilon^{s_0} X) \simeq \hat{Y}(X).$$

Proof. The proof uses a zoom technique introduced by Callot for the Riccati–Hermite equation [5], and presented in a more general case by Benoit in [2]. It consists of a three-step study, the middle step connecting the scales of the first and the last step using a typically nonstandard technique.

Behaviour of $\overline{y}$ *for* $x \ll 0$ *and for* $x \leq x_0 \simeq 0$. As the branch $y = \varphi(x)$ is negative by assumption and thus attracting, as it is preresonant, the solution $\overline{y}$ is defined and satisfies $\overline{y}(x) \simeq \varphi(x)$ for all x such that $x_- \leq x \ll 0$ and thus, by permanence (Fehrele's principle*), there exists $x_0 \simeq 0$, $x_0 \leq 0$, such that this stays true for all $x \in [x_-, x_0]$.

As φ is asymptotic (tangent) to $k|x|^r$ at $x = 0$ with $k \neq 0$, there exists some standard $x'_- \in [x_-, 0)$ such that $\varphi(x) \neq 0$ for all $x'_- \in [x_-, 0)$. As the result is local, without loss of generality, we may assume that $x_- = x'_-$.

As φ is standard, $\varphi(x)$ is appreciable on $[x_-, 0)$ for all $x \not\simeq 0$, and thus $\overline{y}(x) = \varphi(x) + \not{o} = \varphi(x)(1 + \not{o})$ if $x \not\simeq 0$. By Fehrele's principle, there exists some infinitesimal $x'_0 \leq x_0$ such that this stays true for all $x \leq x'_0$. Possibly choosing a smaller infinitesimal x'_0, we may also assume that x'_0/ε^{s_0} is unlimited (negative). Without loss of generality we now change x_0 into $x_0 := x'_0$.

Behaviour for $x \simeq 0$ *outside the* ε^{s_0}*-galaxy of* 0. We now assume that $x_0 \leq x < 0$, $|x|$ large enough for x/ε^{s_0} to be unlimited. We both have to show that $\overline{y}(x)$ is defined and that $\overline{y}(x) = k|x|^r(1 + \not{o})$.

Let $\kappa(x) := \overline{y}(x)/|x|^r$, and choose $k_- < k_+$ standard with the same sign such that $k_- < k < k_+$, and such that $[k_-, k_+]$ contains no other root than k of the algebraic equation ${}_r f_0(-1, K) = 0$. We shall show that it is absurd to assume that $\kappa(x)$ leaves (k_-, k_+) on the external domain under consideration here. This will imply, on one hand, that $\overline{y}(x)$ stays defined, as the compact $\{(x, y) | x_0 \leq x \leq 0, k_-|x|^r \leq \overline{y}(x) \leq$

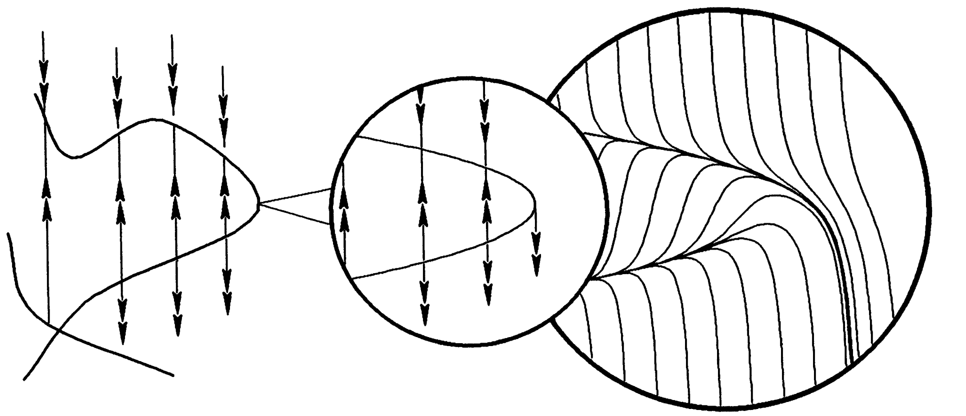

FIG. 5. *The equation* (1), *its image by the microscope* (19), *and its image by the regularizing microscope* (21).

$k_+|x|^r\}(\subseteq \mathrm{hal}\,(0,0))$ is contained in the domain of (1). On the other hand, this implies also that $\kappa(x) \simeq k$ as k_- and k_+ are standard and can be chosen arbitrarily close to k, and so assertion (3.1) will be shown.

So assume that $\kappa(x)$ leaves (k_-, k_+) at $x_1 \geq x_0$, with x_1/ε^{s_0} unlimited, that is, $k_-|x|^r < \overline{y}(x) < k_+|x|^r$ for all $x \in [x_0, x_1)$ and $\overline{y}(x_1) = k_\pm |x|^r$. Let $s_1 (< s_0)$ be such that $x_1/\varepsilon^{s_1} = -1$, and consider the microscope

$$x = \varepsilon^{s_1}\xi, \qquad y = \varepsilon^{rs_1}\eta. \tag{19}$$

Let $\overline{\eta}$ be the image of $\overline{y}$ by this microscope; $\overline{\eta}(\xi) = \varepsilon^{rs_1}\overline{y}(\xi/\varepsilon^{s_1})$ and thus $k_-|\xi|^r < \overline{\eta}(\xi) < k_+|\xi|^r$ for all $\xi \leq -1$ limited, and $\overline{\eta}(-1) = k_\pm$.

This is absurd; indeed, the image of (1) by the microscope (19) is a slow-fast differential equation

$$\varepsilon^d \frac{d\eta}{d\xi} = g(\xi, \eta) \tag{20}$$

with $d := 1 - s_1(1 - r + \mu_r(f_0)) > 0$ and $g(\xi,\eta) := \varepsilon^{-s_1\mu_r(f_0)} f(\varepsilon^{s_1}\xi, \varepsilon^{rs_1}\eta) \simeq {}_r f_0(\xi, \eta)$, the slow curve of which, for $\xi < 0$, is the union of the branches $y = K|x|^r$ for the various roots K of ${}_r f_0(-1, K) = 0$. The differential equation is thus fast at the point $(-1, \overline{\eta}(-1)) = (-1, k_\pm)$, and oriented towards the branch $k|x|^r$, as by asumption $({}_r f_0)'_y\,(-1, k) < 0$, which contradicts that $(\xi, \overline{\eta}(\xi))$ is contained in the crescent $\eta \in [k_-\xi^r, k_+\xi^r]$ for $\xi \leq -1$.

Behaviour for x in the ε^{s_0}-galaxy of 0. We just showed that $\overline{y}(x)$ is defined and satifies the inequalities $k_-|x|^r < \overline{y}(x) < k_+|x|^r$ for all $x \in [x_0, 0]$ such that x/ε^{s_0} is unlimited. Thus by Cauchy's permanence principle* there exists some $x_+ < 0$ such that $X_+ := x_+/\varepsilon^{s_0}$ is limited, and such that this internal property stays true for all $x \in [x_0, x_+]$, which implies, in particular, assertion (3.1).

In other words, the image $\overline{Y}$ of the solution $\overline{y}$ by the microscope

$$x = \varepsilon^{s_0} X, \qquad y = \varepsilon^{rs_0} Y \tag{21}$$

keeps contained, for $X \leq X_+$ limited, in the region $k_-|X|^r \leq \overline{Y}(X) \leq k_+|X|^r$ (Fig. 6) as, for $X \leq X_+$ limited, $x := \varepsilon^{s_0} X \geq x_0$; $\overline{Y}(X)$ is thus limited for all limited

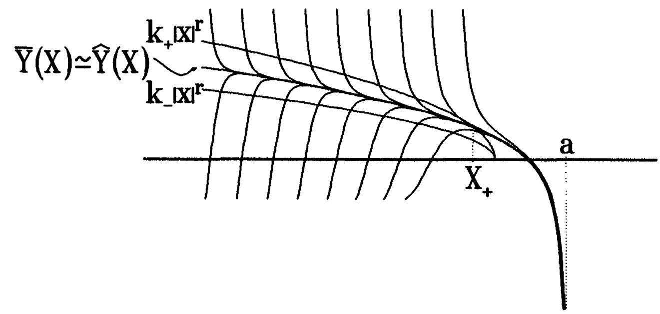

FIG. 6. *The unique river of* (22) *which is contained in the crescent-shaped region* $X < X_+$, $k_-|X|^r \le Y \le k_+|X|^r$ *for all (limited)* X *sufficently negative.*

$X \le X_+$. By the short-shadow lemma, $\overline{Y}$ has thus a shadow $\hat{Y}$, also contained in the close, standard region $k_-|X|^r \le \hat{Y}(x) \le k_+|X|^r$, which is necessarily a solution of the shadow (22) of (1) by the regularizing microscope (21):

$$\frac{dY}{dX} = P(X,Y). \tag{22}$$

But, by Corollary 2.3, one has ${}^rP(X,Y) = {}_rf_0$. As equation ${}^rP(-1,K)$ $(= {}_rf_0(-1,K)) = 0$ has a unique root, k, between k_- and k_+, by Proposition 1.4, $\hat{Y}$ is asymptotic to $k|X|^r$ at $X = -\infty$. As, moreover, $({}^rP)'_Y(-1,k) < 0$, by the rivers-existence theorem [11], $\hat{Y}$ is necessarily the unique river of (22), that is, of type (k,r). As $\hat{Y}$ is the shadow of $\overline{Y}$, one has $\overline{Y}(X) \simeq \hat{Y}(X)$ for all limited X. As k_- and k_+ are standard and of the same sign, $\hat{Y}(X)$ is appreciable for all limited $X \le X_+$, and thus $\overline{Y}(X) = \hat{Y}(X)(1+\not{o})$ for all limited X; hence assertion (3.1). □

Here is, as an example of application, a corollary giving the existence of canards for certain one-parameter famillies of Union-Jack equations. Such canards are not of class S^1 (i.e., the shadow of their image exhibits angles).

COROLLARY 3.2. *Consider a continuous, one-parameter* $d \in \mathbb{R}$ *family of Union-Jack equations*

$$\varepsilon\frac{dy}{dx} = (y-ax)(y-bx)(y-cx) + p(x,y,d) \tag{23}$$

with $a < b < c$ *fixed standard numbers,* p *an internal function with regular* ε*-shadow expansion,* $p(x,y) =: p_0(x,y) + \varepsilon(p_1(x,y,d) + \varepsilon\not{o})$, *on some standard neighbourhood of* $\{(0,0)\} \times [A,C]$, *with* $\mathcal{T}_3(p_0)(x,y) \equiv 0$, p_1 *standard continuous such that* $a = p_1(0,0,A)$, *and* $c = p_1(0,0,C)$.

Then there exist values $d_a \in (A,B)$ *(respectively,* $d_c \in (B,C)$*) of the parameter* d *for which* (23) *has a canard, that is, more precisely, a slow solution, defined on the halo of* 0 *(and further), following, for* $x < 0$, *the attracting branch of a slow-curve tangent to* $y = bx$, *and following, for* $x > 0$, *the repelling branch tangent to* $y = ax$ *(respectively,* $y = cx$*).*

Proof. As the Taylor polynomial of p_0 of degree 3, $\mathcal{T}_3(p_0)$ is zero, the slow curve of (23) has, at $(0,0)$, three negative branches, φ_a^-, φ_b^-, φ_c^-, and three positive branches φ_a^+, φ_b^+, φ_c^+, tangent, respectively, to the three straight lines $y = ax$, $y = bx$, and $y = cx$ (Fig. 7(a)); the three preresonant branches are φ_b^-, φ_a^+, and φ_c^+. We shall show the existence of a value $d_c \in (B,C)$ and of a canard, for $d = d_c$, following φ_b^-

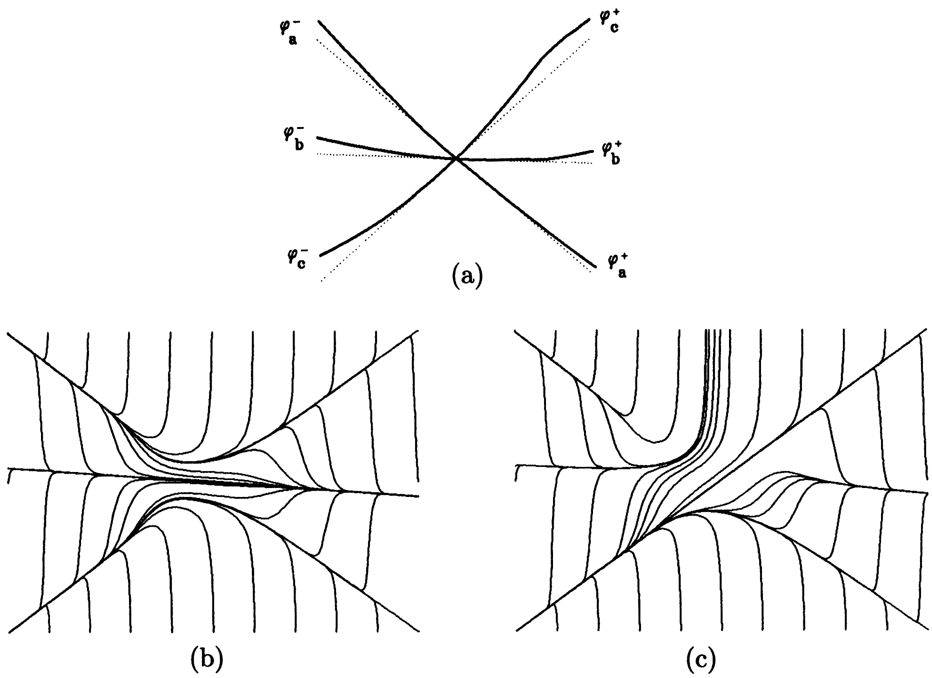

FIG. 7. (a) *the six branches, at* (0,0) *of the (fixed) slow curve of the family* (23); (b) *and* (c) *are some trajectories of* (24), *image by the regularizing microscope of* (23) *for* $d = B$ *and* $d = C$.

and φ_c^+. One would proceed analogously for the existence of a canard following φ_a^+ for some $d = d_a \in (A, B)$.

Let $x_- \ll 0 \ll x_+$ be such that $x_- \in \mathcal{D}(\varphi_b^-)$ and $x_+ \in \mathcal{D}(\varphi_c^+)$. Choose $y_- \simeq \varphi_b^-(x_-)$ and $y_+ \simeq \varphi_c^+(x_+)$; so, (x_-, y_-) and (x_+, y_+) are two "initial conditions" belonging, respectively, to the halo of the two considered branches of slow curve: let $\overline{y}_-$ and $\overline{y}_+$ be the maximal solutions of (23) passing through these two initial conditions; as in (23), $\overline{y}_-$ and $\overline{y}_+$ are dependent continuously on the parameter d.

LEMMA 3.3. *Let* $dY/dX = F(X, Y, d)$ *be a continuous one-parameter* $d \in [D_-, D_+]$ *family of locally Lipschitz differential equations defined for all* $(X, Y) \in [X_-, X_+] \times \mathbb{R}$. *For each* $d \in [D_-, D_+]$, *let* $(X, d) \mapsto \hat{Y}_-$ *and* $(X, d) \mapsto \hat{Y}_+$ *be the maximal solution through* Y_- *and* Y_+ *for* $X = X_-$ *and* $X = X_+$, *respectively* ($Y_\pm$ *may be constant or may depend continuously on the parameter* d). *If there exist* X_0^- *and* X_0^+ *in* $[X_-, X_+]$ *such that* $Y_-(X_0^-, D_-) < Y_+(X_0^-, D_-)$ *and* $Y_-(X_0^+, D_+) > Y_+(X_0^+, D_+)$, *then there exists* $d_c \in [D_-, D_+]$ *such that* $\hat{Y}_-(X, d_c) = Y_+(X, d_c)$ *for all* $X \in [X_-, X_+]$.

Proof. Consider the map $\varphi : [D_-, D_+] \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ that to each value of the parameter d associates the value of $\hat{Y}_-(X_+, d)$ if this value is defined, or $\pm\infty$ if $\hat{Y}_-$ stops to be defined because it tends to $\pm\infty$ for some $X_0 < X_+$. By continuity of F and Y_- with respect to d, φ is continuous. By uniqueness $\varphi(D_-) < Y_+$ and $\varphi(D_+) > Y_+$. Thus, by continuity with respect to d of φ and Y_+, there exists d_c such that $\varphi(d_c) = Y_+$. By uniqueness, for $d = d_c$, $\hat{Y}_- = \hat{Y}_+$. □

Here is the principle of the end of the proof.

Looking at the solutions $\overline{y}_-$ and $\overline{y}_+$ on the regularizing blowup, where, by Theorem 3.1, these solutions are infinitely close to the isolated rivers $\overline{Y}_-$ and $\overline{Y}_+$, shows that for $d = B$ and $d = C$, the solutions $\overline{y}_-$ and $\overline{y}_+$ are defined and slow until $x = 0$, that is, on $[x_-, 0]$ and $[0, x_+]$, respectively, which for $d = B$, one has $\overline{y}_-(0) < \overline{y}_+(0)$, and that for $d = C$, these values at 0 of the solutions are in the inverse order. Thus by Lemma 3.3 there exists some $d_c \in (B, C)$ such that for this value of d, $\overline{y}_- \equiv \overline{y}_+$: this solution is the desired canard. Let us check the various facts stated in this reasoning.

Consider the following regularizing microscope

$$\varepsilon^{1/3}X = x, \qquad \varepsilon^{1/3}Y = y$$

that we already used in example (11). The shadow of the corresponding blowup of (23) is

$$\frac{dY}{dX} = (Y - aX)(Y - bX)(Y - cX) + D, \quad \text{with } D = p_1(0,0,d). \tag{24}$$

Let $\overline{Y}_\pm(X) := \varepsilon^{-1/3}\overline{y}_\pm(\varepsilon^{1/3}X)$ be the images of the solutions $\overline{y}_\pm(x)$. Theorem 3.1 implies that $\overline{Y}_-(X)$ is defined and infinitely close to the unique river $\hat{Y}_-$ asymptotic to $Y = bX$, if X is near-standard in the domain of $\hat{Y}_-$. Analogously $\overline{Y}_+(X)$ is defined and infinitely close to the unique river $\hat{Y}_+$ asymptotic to $Y = cX$ if X is near-standard in the domain of $\hat{Y}_+$.

An elementary study of the two solutions $\hat{Y}_\pm$ of (24) shows that for $D = b$ and $D = c$, these solutions are well defined on an open interval containing $(-\infty, 0]$ and $[0, +\infty)$, respectively.

For $D = b$, $\hat{Y}_-(X) \equiv bX$ is an obvious solution, and $\hat{Y}_+(X) > bX \equiv \hat{Y}_-(X)$, for X positive and large enough, thus for all $X \in \mathcal{D}\hat{Y}_+$, by uniqueness of solutions (Fig. 7(b)). Moreover, as for large values of $Y > 0$, $dY/dX > 0$, solutions of (24) "above" the obvious solution Y_- have to be defined down to $X = -\infty$. So, in particular, $\hat{Y}_+(0)$ is defined and $\hat{Y}_+(0) > \hat{Y}_-(0)$. Now, still for $D = b$, that is, $d = B$, using that $\hat{Y}_+(0)$ and $\hat{Y}_+(0)$ are standard, one has

$$\varepsilon^{-1/3}\overline{y}_-(0) = \overline{Y}_-(0) \simeq \hat{Y}_-(0) \ll \hat{Y}_+(0) \simeq \overline{Y}_+(0) = \varepsilon^{-1/3}\overline{y}_+(0),$$

and thus $\overline{y}_-(0) < \overline{y}_+(0)$.

One uses an analog way of reasoning for $d = C$ (Fig. 7(c)), that is, $D = c$, observing here that $\hat{Y}_+(X) \equiv cX$ is now an obvious solution of (24), and that $\hat{Y}_+ < \hat{Y}_-$. So, for $d = C$, $\overline{y}_-(0) > \overline{y}_+(0)$, which shows the corollary, as explained above. $\square$

Remark. A consequence of Corollary 3.2 is that there exists, for (24) a value $D(= {}^o d)$ for which the unique river asymptotic to bX at $X = -\infty$, and the unique river asymptotic, for example, to aX at $X = +\infty$, are equal. Figure 8 shows a numerical computation of that value. We would be highly interested in an analytical method to determine this value.

3.2. Exiting a critical point's halo. In Theorem 3.1 we considered the question of the image, on a convenient regularizing blowup of any preresonant slow solution. The example considered in Corollary 3.2 can also be used to illustrate the inverse problem, which will be solved below with Theorem 3.4: to the contrary of the canard behaviour of a solution of (23), the "ordinary" behaviour of a preresonant solution that enters, say, for $x < 0$ in the scope of the regularizing microscope, is to stay infinitely close, on the blowup, to a repelling river (at $X = -\infty$). This river solution may also be (see, for example, the left-hand side of Fig. 8) a river at $X = +\infty$ but *attracting* now, so belonging to a one-parameter family of rivers, all with same asymptotic expansion at $X = +\infty$. Such a slow solution crosses the halo of the critical

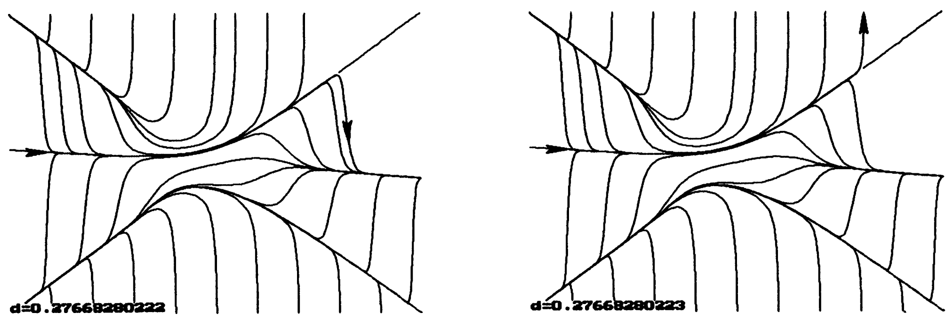

FIG. 8. *Numerical computation of the standard part of the canard value d_c for equation (23), in the case $(a,b,c) = (-1,-0.1,1)$. The behaviour at $X = +\infty$ ("descending" or "ascending" arrow) of the river at $X = -\infty$ of type $(b,1)$ ("left-entering arrow") changes very quickly with the value of d, displayed in the lower left corner.*

point and then follows, for $x > 0$, an attracting branch $\varphi_+ : [0, x_+] \longrightarrow \mathbb{R}$, of type (k_+, r_+), where (k_+, r_+) is precisely the type of the considered river at $X = +\infty$.

Once more, we assume that $(x_0, y_0) = (0,0)$, but that φ is defined on $[0, x_+]$ this time, and is attracting (thus *nonpreresonant*). There is of course an analogous result for any standard (x_0, y_0), and also any repelling branch defined on some "left-interval" $[x_-, x_0]$.

THEOREM 3.4. *Let $k \neq 0$, $r > 0$, $x_+ > 0$ be standard numbers, and $\varphi : [0, x_+] \longrightarrow \mathbb{R}$ be any attracting (k,r)-branch at $(0,0)$ of the slow curve of equation (1); put $s_0 := 1/(1-r+\mu_r(f_0))$. Assume that $({}_r f_0)'_y(-1,k) \neq 0$, and that (1) is a regular deformation for the growth-type of the branch φ of the simplified equation*

$$\varepsilon \frac{dy}{dx} = f_0(x,y).$$

Denote by $P(X,Y)$ the polynomial such that the shadow of the blowup of (1) by the regularizing microscope

$$x = \varepsilon^{s_0} X, \qquad y = \varepsilon^{r s_0} Y \tag{25}$$

is equal to

$$\frac{dY}{dX} = P(X,Y). \tag{26}$$

Under these assumptions, (26) has an attracting river $\hat{Y} : (a, +\infty)$ of type (k,r) at $X = +\infty$.

Let $\overline{y}$ be any maximal solution of (1), and let $\overline{Y}(X) := \varepsilon^{-r s_0} \overline{y}(\varepsilon^{s_0} X)$ be its image by the regularizing microscope (25). Let $X_0 \gg a$ be limited, and define $x_0 = \varepsilon^{s_0} X_0$.

(1) *If $\overline{Y}(X_0) \simeq \hat{Y}(X_0)$, then for any X near-standard in $]a, +\infty[$ and for $x := \varepsilon^{s_0} X$, $\overline{y}(x)$ is defined and satisfies*

$$\varepsilon^{-r s_0} \overline{y}(\varepsilon^{s_0} X) \; (= \overline{Y}(X)) \simeq \hat{Y}(X).$$

(2) *For all $x \in [0, x_+]$ such that $x \simeq 0$, if x/ε^{s_0} is unlimited, then $\overline{y}(x) = k x^r (1 + \phi)$.*

(3) *For any appreciable $x \in]0, x_+]$, $\overline{y}(x) \simeq \varphi(x)$.*

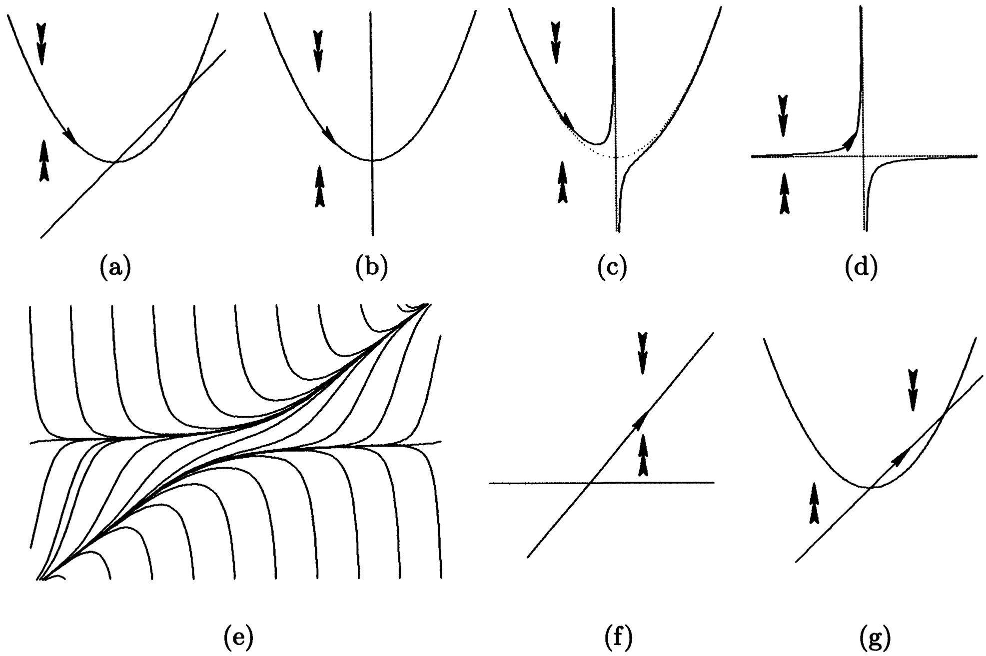

FIG. 9. *The passage through the halo of the critical point of* (27). (a) *and* (g): $\varepsilon^s = 1$, *the initial scale*; (b): $\varepsilon^s \simeq 0$ *and* $\varepsilon/\varepsilon^{3s} \simeq 0$; (c): $\varepsilon^{3s} = \varepsilon$, *one puts* $\eta = \varepsilon^{1/3}$; (c)–(e): *the microscopes of Theorem* 3.1 *in the case* $0 > r$ $(= -1)$; (e)–(g): *the microscopes of Theorem* 3.4 *applied to the positive branch* $y = x$.

Proof. By Corollary 2.3, we have ${}^rP(X,Y) = {}_rf_0$. By Proposition 1.2, we have ${}^rP(1,k) = {}_rf_0(1,k) = 0$, and, as the branch φ is attracting, $({}_rf_0)'_y(1,k) < 0$. Thus by the rivers existence theorem, there exists an attracting river $\hat{Y}$: $(a, +\infty) \longrightarrow \mathbb{R}$, of type (k,r). The short-shadow lemma implies that the solution $\overline{Y}$ of blowup of (1) by the regularizing microscope (25) is defined for all X near-standard in $]a, +\infty[$, and that $\overline{Y}(X) \simeq \hat{Y}(X)$; hence assertion (3.4).

Assertions (2) and (3) are obtained by reasoning with the same microscopes as in the proof of Theorem 3.1; the existence of the slow solution up to x_+ follows from the attractivity of the slow curve at each of these scales. □

3.3. Singular deformations. We shall raise the question of equations with singular deformations for some branches of the slow curve on an example, and more precisely on the example of (11) which we already used to introduce that notion. This will also give us the oportunity to sketch how to deal with slow curves at some x_0 with branches of type (k,r), with $r < 0$. For the sake of simplicity, we do not consider here this question in the general case (that is, only relevant $f_0(x,y)$ that are polynomials in y^α).

Recall that (11) is equation

$$\varepsilon\frac{dy}{dx} = (y - x^2)(x - y) + \varepsilon(a + bx) \tag{27}$$

for which the term $\varepsilon(a+bx)$ is a singular deformation for the branch $y = x^2$ for any standard $a \neq 0$. This branch being preresonant for $x < 0$, we shall, more precisely, be interested in the behaviour of any preresonant solution $\overline{y}$ with initial condition

(x_-, y_-) such that x_- is appreciable and negative and $y_- \simeq x_-^2$. The branch $y = x^2$ being attracting, $\overline{y}(x)$ is defined and $\overline{y}(x) \simeq x^2$ for all x such that $x_- \leq x \ll 0$, and thus, by Fehrele's principle, for some $x \simeq 0$. As soon as $x \simeq 0$, we study $\overline{y}$ using the "zoom" (microscope with strengh varying with s)

$$\varepsilon^s X = x, \qquad \varepsilon^{2s} Y = y. \tag{28}$$

Let $\overline{Y}_s$ be the image of $\overline{y}$ by this microscope; thus it is a solution of equation

$$\frac{dY}{dX} = \varepsilon^{-1-s}[\varepsilon^{3s}(Y - X^2)(X - \varepsilon^s Y) + \varepsilon(a + bX)]. \tag{29}$$

As long as $s \ll \frac{1}{3}$, and more precisely as long as $\varepsilon/\varepsilon^{3s} \simeq 0$, (29) is slow-fast, with slow curve $X(Y - X^2)$, which is attracting for $X < 0$ (Fig. 9). Using the same reasoning as in the proof of Theorem 3.1, we see that for all these values of s and all appreciable $X < 0$, $\overline{Y}_s(X) \simeq X^2$, and thus $\varepsilon^{-2s}\overline{y}(\varepsilon^s X) \simeq X^2$.

Increasing the strengh of the zoom, that is, increasing s such that the ratio $\varepsilon^{3s}/\varepsilon$ becomes appreciable, say equal to 1, that is $s = \frac{1}{3}$, then $\overline{Y}_{1/3}$ is a solution of

$$\eta\frac{dY}{dX} = X(Y - X^2) + a + \eta(bX - Y(Y - X^2)), \tag{30}$$

with $\eta := \varepsilon^{1/3}$; this is still a slow-fast equation, but a change occured in the slow curve (Fig. 9(c)): it's no longer a parabola, but the curve $X(Y - X^2) + a = 0$, which, as $a \neq 0$, is now only asymptotic to the parabola $\{Y = X^2\}$ at $X = -\infty$; but at $X = 0$, it is asymptotic to $Y = -a/X$. *This part of the slow curve has to be seen as a (negative) branch φ of type $(a, -1)$ at $(0, 0)$, which means asymptotic to $a|X|^r$, with $r = -1$.*

Notice that now the term ηbX is a *regular* deformation term with respect to the preresonant branch φ, of type $(a, -1)$, for the simplified equation $\eta(dY/dX) = X(Y - X^2) + a$, for which the "microscope" (it is indeed a microscope with respect to X, but it is a *macroscope* with respect to Y)

$$\eta^{\frac{1}{2}}\mathbf{X} = X, \qquad \eta^{-\frac{1}{2}}\mathbf{Y} = Y \tag{31}$$

changes (30) into a regular perturbation for the polynomial differential equation with rivers

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = -\mathbf{Y}^2 + \mathbf{X}\mathbf{Y} + a, \tag{32}$$

which exhibits in particular a river $\hat{\mathbf{Y}}_-$ of type $-a/\mathbf{X}$ at $\mathbf{X} = -\infty$. Observe now that in Theorem 3.1, we assumed that r, the growth type of the branch, is positive; this was done for the sake of simplicity, and also for the sake a generality with respect to the equation (which was assumed to be $\mathcal{C}^\infty$ and not only a polynomial), a negative growth type at x_0 making no sense in the $\mathcal{C}^\infty$ case. One checks easily that we only used that $r \neq 0$ (and thus $kr \neq 0$). So we come to the same conclusion here and see that for limited $\mathbf{X}$ sufficiently negative,

$$\hat{\mathbf{Y}}_-(\mathbf{X}) \simeq \overline{\mathbf{Y}}_{\frac{1}{3}}(\mathbf{X}),$$

where $\overline{\mathbf{Y}}_{1/3}$ denotes the image of the solution $\overline{Y}_{1/3}$ on the blowup using the microscope (31).

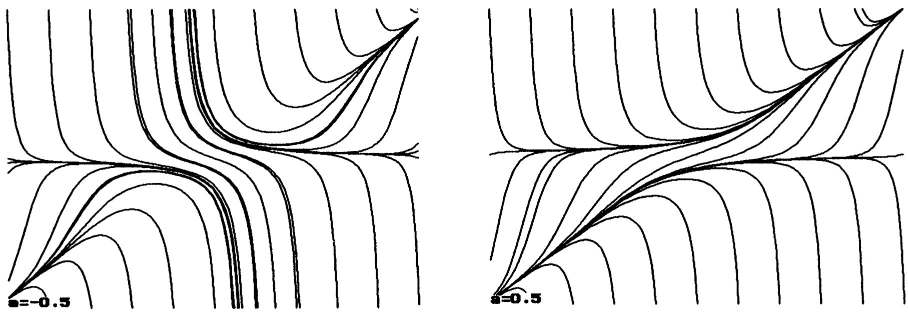

FIG. 10. *The blowup of equation* (27) *by the regularizing microscope* (31) *for some negative and some positive value of the parameter a.*

It is easy to study the behaviour of the rivers of (32) (see Fig. 10). One shows in particular that for $a > 0$, the river $\hat{\mathbf{Y}}_-$ at $\mathbf{X} = -\infty$ is also a river at $\mathbf{X} = +\infty$, but attracting and of type $(1, 1)$ this time.

The time has come to express the microscope we obtained by the two successive microscopes we applied, namely, $x = \varepsilon^{1/3}X$, $y = \varepsilon^{2/3}Y$, and $X = \varepsilon^{2/3}\mathbf{X}$, $Y = \varepsilon^{-1/3}\mathbf{Y}$, as $\eta = \varepsilon^{1/3}$. Finally, we have

$$x = \varepsilon^{\frac{1}{2}}\mathbf{X}, \qquad y = \varepsilon^{\frac{1}{2}}\mathbf{Y} :$$

we merely resulted in the regularizing microscope relative to the branch of type $(1, 1)$ (with respect to which the complete equation (27) is a regular deformation of the simplified equation). So we can apply Theorem 3.4 for $\mathbf{X} > 0$, and we get (Fig. 9(e)–9(g)) that for $a > 0$ (standard), the solution $\overline{y}$ stays slow for $x > 0$, and follows the attracting curve $y = x$.

Let us observe that for $a < 0$ (standard), Fig. 10(b) suggests what the behaviour of $\overline{y}$ will be for $x \gg 0$: when $\mathbf{X}$ increases, $\hat{\mathbf{Y}}_-$ decreases faster than any linear function and thus, coming back to the initial scale, $\overline{y}$ jumps towards y unlimited negative.So we can deduce that $\overline{y}$ will assume, for some $a \simeq 0$, intermediate behaviours, and especially that for any limited b, there exists some $a(b)$ such that $\overline{y}_{a(b),b}$ is a canard of (27).

This example shows how to reduce a singular deformation problem: r being chosen with respect to the preresonant branch φ followed by the considered solution, one increases gradually the value of s in the zoom (28): for small values of $s > 0$, at that scale the solution follows the slow curve equal to the growth type of φ. For some smallest standard value $s_1 (< s_0$, unless the deformation would just be regular), one or several terms in the deformation get the same size order as those of f_0: Proposition 1.2 shows that such terms can only rise from those f_p such that $1 \le p < 1 + s_0(r-1)$. One gets a new slow-fast equation ("ε" becomes "η"), the standard part of which being henceforth a polynomial, and the slow curve being an *algebraic curve*, which may exhibit new critical points and which may once more be analysed in branches at these critical points, but also branches at $X = \pm\infty$, or $Y = \pm\infty$ $(r < 0)$. Once the critical points with finite abscissa have been found (this is the only difficult operation), one can study theses branches with convenient Newton polygons. If the slow curve at the initial scale exhibits a preresonant branch of type (k, r) for $\pm(x - x_0) > 0$, the slow curve of the blowup will exhibit a branch of type (k, r) at $X = \pm\infty$ such that the solution will follow up to some new critical point of finite abscissa and finite or infinite ordinate: it suffices to reapply the same process; as already mentioned, the

Theorems 3.1 and 3.4 apply in both cases, and this makes it possible to "follow" the considered solution, at least when no canard situations occur, which would imply an additional canard study. We do not doubt that this process will finally result (if no nonzero occuring function is infinitely flat at some of the critical points, which will not happen if the functions are assumed to be analytic) in a regularizing blowup, but it would of course be interesting to solve this question in an effective manner (symbolic machine computations). We will not consider here this purely algebraic question that involves only the equation, our goal being to show the role played by rivers in the behaviour of the slow *solutions.*

4. Appendix. Let us briefly sketch the nonstandard tools involved in this paper.

The vocabulary of infinitesimals had gradually been forsaken by most mathematicians because of the contradictions that this terminology seemed to introduce. To overcome these contradictions, logicians like Löwenheim and Skolem introduced so-called "nonstandard models," mathematical constructions that would, for example, add "infinitely large elements" to the integers. Abraham Robinson extended these techniques in order to build efficient nonstandard models that contain all of what is needed by mathematical analysis (Banach spaces, measure spaces, etc.) He also pointed out that his construction should indicate how to use safely the infinitesimal terminology within the usual mathematical sets, like $\mathbb{R}$, the main idea being not to apply blindly classical results to "external sets" like the set of standard integers. It is Nelson that achieved this program, in his October 1977 paper in the Bulletin of the American Mathematics Society that indicates what to do (and what *not* to do) using that terminology. Actually, Nelson's Internal Set Theory introduces just one new word, *standard*, and gives the rules for correct use of it.

We use Nelson's approach (and thus the usual real line, not some elaborate model); here is how it works. Every classical object is standard or not: we just need tools to determine which is which. Any uniquely defined object (possibly using other objects that have already been shown to be standard) is standard. So $\emptyset$, 0, 1, π, sin, $\mathbb{R}$, $\mathcal{C}_0(\mathbb{R})$ are standard. An equivalent statement, called *transfer*, asserts that classical-type theorems are true if and only if they are true for standard elements. Things become more interesting with the theorem asserting that any infinite set contains nonstandard elements (infinite means, as usual, in one–one correspondence with one of its proper subsets). So $\mathbb{N}$ has nonstandard elements, and any nonstandard integer, say ω, is larger than any standard one. So we have an infinitely large number ω. Taking its inverse (in $\mathbb{Q}$ or $\mathbb{R}$), we get a nonzero *infinitesimal* $\varepsilon = 1/\omega$, i.e., $|\varepsilon|$ is smaller than $1/n$ for all standard $n > 0$. A number is *limited* if its absolute value is smaller than some standard number (one should keep the word "finite" for questions of cardinality: the set $\{1, \ldots, \omega\} \subset \mathbb{N}$ is finite, but its cardinal is infinitely large, or better: *unlimited*). It is *appreciable* if it is neither infinitesimal nor unlimited. Here are two notations: ϕ (pronounce *zerobar*) will always denote an infinitesimal and $\pounds$ a limited number, but two occurences of ϕ are *not* necessarily equal, and analogously for $\pounds$. Please notice the difference between the empty-set symbol $\emptyset$ and this symbol ϕ. We write $x \simeq y$ if and only if $x - y$ is infinitesimal, and, to the contrary, $x \ll y$ if and only if $x - y$ is not infinitesimal.

As $\mathbb{R}$ is complete, any limited real number l is infinitely close to a standard one, called its standard part, and denoted by ${}^{o}l$. This extends easily to any standard finite-dimensional Banach space B. For any standard $C \subseteq B$, if ${}^{o}l \in C$, we say that l is near-standard in C. If $\gamma \subseteq B$ has all its limited points near-standard in C, it is *infinitely close to C*. If γ is a curve (think of a trajectory), we also say that γ

follows C. If, moreover, C is standard and any of its standard points is infinitely close to some point of γ, then C is called the *shadow* of γ, and we write ${}^o\gamma = C$ (actually, on any standard compact subset, it *is* a standard part for the Haussdorf metric). A standard function f_0 is the shadow of the function f if their graphs are also. We use this terminology for differential equations, just identifying the equation $y' = f(x, y)$ with the function f. Given a differential equation $y' = f(x, y)$ and its shadow $y' = {}^of(x, y)$, both assumed to be locally Lipschitz, the *short-shadow lemma* [9, Thm. 8.2.2] indicates the relationship existing between the shadow of some segments of solution of the initial equation and a standard solution of the shadow equation. The segment should be short in the sense that the difference between two elements in its domain has to be limited. The lemma also gives existence results for solutions of the initial equation from existence assumptions on the shadow equation.

To make it possible to assume that $\varepsilon > 0$ is a *fixed* infinitesimal is a key point in the use of nonstandard analysis to singular perturbation theory. For example, dealing with the question of *existence*, the fact that solutions have their values in a (finite-dimensional and thus) locally compact space, one can refer to the prolongation theorem, which ensures that the maximal solution leaves any closed subset of the domain of the equation; it might be more cumbersome to deal with an (ad hoc) infinite-dimensional space (of $\mathbb{R}^2$-valued functions, defined for $\varepsilon \in (0, \varepsilon_0)$). A still more important reason is that it makes it easier to let other parameters vary. Expansions involving ε may enter "by themselves" for a *number* a, as, for example, for the necessary value $a = 1 - \frac{1}{8}\varepsilon - \frac{3}{32}\varepsilon^2 + \varepsilon^2\phi$ of some parameter (for the existence of canards [3]).

A number a has an ε-shadow expansion, that we shall write $a = \sum a_n\varepsilon^n$, if $(a_n)_{n\geq 0}$ is a standard sequence of numbers and for any *standard* n, $a = a_0 + a_1\varepsilon + \cdots + a_n\varepsilon^n + \varepsilon^n\phi$ (this formal expansion usually does not converge for any nonzero value of ε). A function f has an ε-shadow expansion, write $f = \sum f_n\varepsilon^n$, if $(f_n)_{n\geq 0}$ is a standard sequence of functions on some domain D, and for any *standard* n, and any x near-standard in D, $f(x) = f_0(x) + f_1(x)\varepsilon + \cdots + f_n(x)\varepsilon^n + \varepsilon^n\phi$. The expansion is *regular* if the functions f_n are $\mathcal{C}^\infty$, and for any *standard* p, $f^{(p)} = \sum f_n^{(p)}\varepsilon^n$. For example, if F is standard and $\mathcal{C}^\infty$ with respect to (x, e) and if $\varepsilon > 0$ is some infinitesimal, then Taylor's theorem shows that for $f(x) := F(x, \varepsilon)$, one has $f = \sum F_{e^n}^{(n)}(., \varepsilon)\varepsilon^n/n!$. But there exist useful examples which are not of that kind, for example, $f(x) = \Phi(x, \varepsilon, a)$, $\Phi \in \mathcal{C}^\infty$ standard, and a is equal to the sum of the smallest term of some diverging expansion, as for the canard values of the Van der Pol equation [3]. If f has a regular ε-shadow expansion, and $\overline{y}$ is a slow solution of $\varepsilon y' = f(x, y)$ that follows a branch of the slow curve $\{{}^of(x, y) = 0\}$ with no critical point, then $\overline{y}$ has also an ε-shadow expansion, and this expansion is the same for all slow solutions following the same branch [8].

It is possible to deal with external sets, provided one takes care not to apply (blindly) classical theorems to them. We say that a set (i.e., a subcollection defined using the extended language of some classical set) is *external*, if some classical result is wrong for it. The set ${}^{st}\mathbb{N}$ of standard integers is external (it is bounded by ω but has no least upper-bound), and so is the *principal galaxy* $\mathbb{G}$ *of* $\mathbb{R}$, the set of all limited real numbers (the previous example could be deduced from it just by intersecting it with the standard set $\mathbb{N}$), or the α-galaxy $x_0 + \alpha\mathbb{G}$ of any x_0 for any $\alpha \neq 0$. One has specific results for external sets, which can be used as *permanence* principles (or overspill principles). The most obvious one is the "Cauchy principle" (named after Cauchy's original statement on continuity of the limit of a sequence of continuous functions) that just states that an external set is not internal. So, an internal property cannot hold only on an external set (otherwise this statement would define the external set, and there would be a contradiction in classical mathematics) and the property must overspill to other points. An external set is a *halo* if it is the *intersection* on all standard indices of an internal family of sets (think of $\mathrm{hal}(0) := \bigcap_{^{st}\mathbb{N}}[-\frac{1}{n}, \frac{1}{n}]$), and it is a *galaxy* if it is the *union* on all standard indices of such a family (think of $\mathbb{G} := \bigcup_{^{st}\mathbb{N}}[-n, n]$). So the external domain where a function is limited or appreciable is a galaxy, and the external domain where it is infinitesimal or unlimited is a halo. Fehrele's principle states that no halo is a galaxy, so, for example, two functions cannot be infinitely close to each other only on a galaxy. This is a generalization of an easy but nice result called Robinson's lemma. The typical use of this permanence result is when it is necessary to overspill an "up to an infinitesimal" estimate from all limited values of the variable up to some infinitely large ones: the domain where this type of estimate holds is a halo, whereas the external set of limited numbers is a galaxy; so the halo must be strictly larger than this galaxy.

We refer to [9] for proofs, further results, and bibliography.

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