PERSISTENCE OF STABILITY LOSS FOR DYNAMICAL BIFURCATIONS. I

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Many problems in oscillation theory lead to differential-equation systems with fast and slow variables [1]. We assume that, for fixed values of the slow variables, the system with fast variables has an equilibrium position or a limit cycle, and that the drift of the slow variables leads to a dynamical bifurcation: The equilibrium position (the cycle) loses its stability but remains nondegenerate. For the equilibrium a pair of conjugate eigenvalues leaves the left half-plane without passing through 0; for a cycle the multipliers leave the unit circle without passing through 1. We prove that, in analytic systems, the stability loss persists: The phase points remain near the unstable equilibrium (the cycle) as long as the slow variables do not approach values of order 1. The rate of change of the slow variables can be arbitrarily small. Such a persistance of stability loss is not in general found in nonanalytic systems: A cut-off from the unstable equilibrium (cycle) takes place near the bifurcation values of the slow variables.

Persistence of an equilibrium-position stability loss was first noted in [2], for a model equation system, where the asymptotic behavior of the time of equilibrium cut-off was calculated. In [3, 4], results in [2] were extended to apply to some special system classes. The results described in the present work were announced in [5, 6]. In [5] we also gave a lower bound for the equilibrium cut-off time, and asymptotic properties of the cut-off time for several cases. The establishment of this bound will be described in the second part of this article.

1. Fast-Variable Systems. Here fast-variable systems are understood to be differential-equation systems of the following form, often encountered in oscillation theory [1]:

$$\dot{x}=f(x,\ y,\ \varepsilon),\quad x\in R^n,$$
$$\dot{y}=\varepsilon g(x,\ y,\ \varepsilon),\quad y\in R^m. \tag{1.1}$$

Here ε is a small parameter, x denotes fast variables, and y denotes slow variables. The right side in (1.1) is at least twice continuously differentiable with respect to (x, y) and once with respect to ε for $(x, y)\in D$ and $|\varepsilon| < \varepsilon_1$, where D is a domain in R^{n+m} and ε_1 = const >

In the first equation in (1.1), for x, the variable y is assumed to be constant and $\varepsilon = 0$. We consider two cases: a) the fast system has an equilibrium position that is nonsingular and depends continuously on y; b) the fast system has a limit cycle that is nonsingular and depends continuously on y. The equilibrium position and the cycle, imbedded in R^{n+m}, are designated L_y and for brevity will be called "states."

A slow system is a system $\dot{y} = \varepsilon G(y)$ in which the function G(y), for an equilibrium, is the value of g(x, y, 0) at the equilibrium and, for a cycle, is the time-mean of the values of g(x, y, 0) on the cycle in the fast motion.

2. Persistence of Stability Loss. Fix a solution of the slow system

$$y=Y(\tau),\ \tau=\varepsilon t\in[\tau_0,\ \tau_1],\ Y(\tau_0)=y_0,\ \varepsilon>0,$$

having no common points with the boundary of the projection of D on the y-space. Let the "state" $L_{Y(\tau)}$, in the fast system, be asymptotically stable in the linear approximation for $\tau\in[\tau_0,\ \tau_*)$, where $\tau_*\in(\tau_0,\ \tau_1)$. For $\tau = \tau_*$ this stability is lost. In the case of equilibrium for $\tau = \tau_*$, there is a pair of conjugate imaginary eigenvalues and 0 is not an eigenvalue; in the case of a cycle for $\tau = \tau_*$, there is either a pair of conjugate multipliers on the unit circle or the multiplier −1 but not the multiplier 1. The following theorem describes the phenomenon of stability-loss persistence.

All-Union Medical-Instrument-Manufacture Research Institute. Translated from Differentsial'nye Uravneniya, Vol. 23, No. 12, pp. 2060-2067, December, 1987. Original article submitted July 7, 1986.

with respect to ε, then the phase point $[x(t), y(t)]$ of this system, whose motion begins for $\varepsilon t = \tau_0$ and $y(t) = y_0$ in a C_1^{-1}-neighborhood of the "state" L_{y_0}, will be in the $C_3\varepsilon$-neighborhood of the "state" $L_{Y(\varepsilon t)}$ for $\varepsilon t \in [\tau_*, \tau_* + C_2^{-1}]$.

Here the C_i (and the c_i and k_i below) are sufficiently large positive constants, i.e., they do not depend on ε. The appearance of C_i, c_i, and k_i in the text in a relation is equivalent to the assertion that there are constants satisfying this relation for sufficiently small $\varepsilon > 0$.

This theorem is proved in Sec. 3. It is clear from this theorem that the phase point is in the domain indicated also for $\varepsilon t \in [\tau_0 + C_4 \varepsilon |\ln \varepsilon|, \tau_*]$. The theorem is known for $\varepsilon t \in [\tau_0 + C_4 \varepsilon |\ln \varepsilon|, \tau_* - C_5^{-1}]$ ([7], p. 55 and [8]).

The equilibrium-stability-loss persistence is shown graphically in Fig. 1. It is assumed that fast motion loses its stability weakly, i.e., for $\tau = \tau_* + 0$ the equilibrium corresponds to a stable limit cycle of the fast system, whose size (the amplitude of the self-oscillations) increases like $\sqrt{\tau - \tau_*}$ ([9], p. 496). In the fast-slow system, the equilibrium cut-off and the transition to the cycle occurs for $\tau > \tau_* + C_2^{-1}$, when the cycle already is of dimension of order 1. The oscillation amplitude increases discontinuously (with a jump of order $|\ln \varepsilon|$ at time t) from the lower order ε to order 1, i.e., the stability loss appears to be produced stably.

The stability-loss persistence leads to hysteresis: If τ (Fig. 1) is a parameter varying periodically, then the self-oscillations arise suddenly for $\tau > \tau_* + C_2^{-1}$ and disappear smoothly for $\tau \approx \tau_*$.

In the consideration of the physical importance of the phenomena described, it must be taken into account that they are annihilated by weak random noise. Theorem 1 (Sec. 3) shows that if, for $\tau = \tau_*$, the phase point moves by an amount $\delta \ll \varepsilon$, then it can remain in an $O(\varepsilon)$-neighborhood of the state under consideration only for a time generally of order $\sqrt{\varepsilon^{-1} \ln(\varepsilon/\delta)}$. The persistance will be observed only if the noise is exponentially small with respect to ε.

3. Proof of Theorem 1. We give the proof for the case in which L_y is a cycle. If L_y is in equilibrium, then the first step of the proof, which is the transition from a cycle to an equilibrium state of a system with periodic coefficients (Sec. 3.1), can be omitted. The remaining steps only require the obvious simplification. We write $|\cdot|$ for the Hermitian norm.

3.1. A Preliminary Transformation. Let φ be the angular coordinate on the cycle L_y varying uniformly in fast motion. A point in the neighborhood of the cycle can be located by the coordinates $\varphi \pmod{2\pi}$ and $z \in R^{n-1}$, where z is the rectangular coordinate in the area transversal to the cycle passing through the point of the cycle with angular coordinate φ, on the cycle $z = 0$. After the substitution $x, y \mapsto z, \varphi$, the function y becomes smooth, and analytic where the system is analytic. In the new variables the fast system in (1.1) becomes

$$\dot{z} = a(\varphi, y)z + O(|z|^2) + O(\varepsilon), \quad \dot{\varphi} = \omega(y) + O(|z|) + O(\varepsilon), \quad \dot{y} = \varepsilon b(\varphi, y) + \varepsilon O(|z|) + O(\varepsilon^2). \tag{3.1}$$

3.2. A Lemma Concerning an Exponential Small Discrepancy. LEMMA 1. In the complex c_1^{-1}-neighborhood of the circle $z = 0$, $y = y_* = Y(\tau_*)$, a real-analytic variable change $z = Z + U(\varphi, y, \varepsilon)$, 2π-periodic as a function of φ, and differing by $O(\varepsilon)$ from the identity variable change, transforms the first equation of the system (3.1) into

$$\dot{Z} = a(\varphi, y)Z + \varepsilon O(|Z|) + O(|Z|^2) + O(\exp(-c_2^{-1}/\varepsilon)). \tag{3.2}$$

Proof. The required variable change is constructed as a composition of a large number, of order $1/\varepsilon$, of successively determined substitutions. Under these substitutions, the free term in the equation for z (not vanishing for $z = 0$) decreases at least as fast as the terms of a geometric series. The bounds differ only slightly from those in [10-12].

After j substitutions, the equation system becomes

$$\begin{gathered} \dot{z} = a(\varphi, y)z + \Phi_j(\varphi, y, z) + h_j(\varphi, y), \\ \dot{\varphi} = \omega(y) + \Psi_j(\varphi, y, z), \quad \dot{y} = \varepsilon\Theta_j(\varphi, y, z), \\ \Phi_j = \varepsilon O(|z|) + O(|z|^2), \quad h_j = O(\varepsilon), \quad \Psi_j = O(|z|) + O(\varepsilon), \quad \Theta_j = O(1). \end{gathered} \tag{3.3}$$

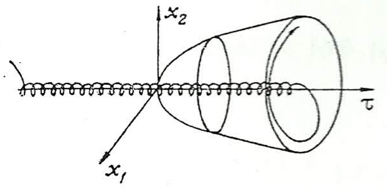

Fig. 1

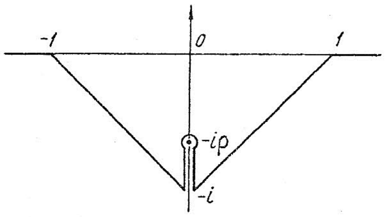

Fig. 2

brevity we omit the argument ε in Φ_j, Ψ_j, and Θ_j. The following $(j+1)$-th variable change $z = Z + u_{j+1}(\varphi, y)$, where u_{j+1} is a solution, 2π-periodic in φ, of the following inhomogeneous stem with periodic coefficients:

$$\frac{du}{d\varphi} = \omega^{-1}(y)\,a(\varphi, y)\,u + \omega^{-1}(y)\,h_j(\varphi, y), \quad y = \text{const}.$$

$|y - y_*| < c_1^{-1}$, the corresponding homogeneous system has no multiplier 1. The required solution is thus

$$u_{j+1} = \omega^{-1}R(\varphi, y)[E - R(2\pi, y)]^{-1}R(2\pi, y)\int_0^{2\pi} R^{-1}(\vartheta, y)h_j(\vartheta, y)\,d\vartheta + \omega^{-1}R(\varphi, y)\int_0^{\varphi} R^{-1}(\vartheta, y)h_j(\vartheta, y)\,d\vartheta, \quad (3.4)$$

re $R(\varphi, y)$ is a fundamental solution matrix of the homogeneous system and $R(0, y) = E$; here is the unit matrix. For Z we again obtain a system of the form (3.3), and

$$h_{j+1} = \Phi_j(\varphi, y, u) - \frac{\partial u}{\partial \varphi}\Psi_j(\varphi, y, u) - \varepsilon\frac{\partial u}{\partial y}\Theta_j(\varphi, y, u), \quad u = u_{j+1},$$

$$\Phi_{j+1} = \Phi_j(\varphi, y, Z+u) - \frac{\partial u}{\partial \varphi}\Psi_j(\varphi, y, Z+u) - \varepsilon\frac{\partial u}{\partial y}\Theta_j(\varphi, y, Z+u) - h_{j+1}, \quad (3.5)$$

$$\Psi_{j+1} = \Psi_j(\varphi, y, Z+u), \quad \Theta_{j+1} = \Theta_j(\varphi, y, Z+u).$$

e order of the free term with respect to ε has increased to 1. However further substitutions lead to divergence, so that we cannot eliminate the free term. Our further calculations e to show that it can be made exponentially small.

We now obtain bounds. The system (3.1) was obtained in $D_0 = \{z, \varphi, y : |z| < k_1^{-1}, |\mathrm{Im}\,\varphi| < k_1^{-1}, |y - y_*| < k_1^{-1}\}$. In the domain $D_3 = D_0 - k_1^{-1}/4$† we have found the compositions in the first ee substitutions discussed above, which decrease the free term to the order $O(\varepsilon^4)$. The stem is reduced to the form (3.3) with $j = 3$ and

$$|h_3| < k_2\varepsilon^4, \quad \left|\frac{\partial\Phi_3(\varphi, y, 0)}{\partial z}\right| < k_2\varepsilon, \quad \left|\frac{\partial^2\Phi_3}{\partial z^2}\right| < k_2,$$

$$|\Psi_3(\varphi, y, 0)| < k_2\varepsilon, \quad |\partial\Psi_3/\partial z| < k_2, \quad |\Theta_3| < k_2, \quad (3.6)$$

$$|\omega| > k_2^{-1}, \ |R| < k_2, \ |R^{-1}| < k_2, \ |(E - R(2\pi, y))^{-1}| < k_2.$$

bsequent bounds are obtained by induction. Let $M_j = 2^{-(j-3)}k_2\varepsilon^4$ and assume that subsequent i e variable and, for $3 \leqslant j \leqslant i$, we have

$$|h_j| < M_j, \quad \left|\frac{\partial\Phi_j(\varphi, y, 0)}{\partial z}\right| < 2k_2\varepsilon, \quad \left|\frac{\partial^2\Phi_j}{\partial z^2}\right| < 2k_2, \quad (3.7)$$

$$|\Psi_j(\varphi, y, 0)| < 2k_2\varepsilon, \quad |\partial\Psi_j/\partial z| < k_2, \text{ and } |\Theta_j| < k_2.$$

$D_j = D_3 - (j-3)K\varepsilon$. Here $K > 1$ is an as yet undetermined positive constant. We must ove that (3.7) holds for $j = i + 1$ if D_{i+1} is not empty. The last two inequalities in (3.7) e obvious, since Ψ_j and Θ_j are only Ψ_3 and Θ_3, expressed in terms of the new variables.

Consider the $(j+1)$-th variable changes, $j = 3, \ldots, i$. It follows from (3.4), (3.6), d (3.7) that $|u_{j+1}| < k_3M_j < \varepsilon K/4$. For $(Z, \varphi, y) \in D_j - K\varepsilon/2$ the variable change $z = Z + u_{j+1}$ is ade, and $(z, \varphi, y) \in D_j - K\varepsilon/4$. From Cauchy's inequality [10], (3.5), and (3.7) it follows that e following inequalities hold for $(Z, \varphi, y) \in D_j - K\varepsilon/2$:

† We write D for the ρ-set of points whose ρ-neighborhoods are in D.

$$\left|\frac{\partial\Phi_{j+1}}{\partial Z}-\frac{\partial\Phi_j}{\partial Z}\right|<\frac{k_8M_j}{(K\varepsilon)^2},\quad \left|\frac{\partial^2\Phi_{j+1}}{\partial Z^2}-\frac{\partial^2\Phi_j}{\partial Z^2}\right|<\frac{k_8M_j}{(K\varepsilon)^3}.$$

Putting $K = 2k_5$, we find that $|h_{i+1}|<M_i/2=M_{i+1}$,

$$|\partial\Phi_{i+1}(\varphi, y, 0)/\partial Z|<|\partial\Phi_3(\varphi, y, 0)/\partial Z|+\frac{k_9}{\varepsilon^2}\sum_{j=3}^{i}M_j<k_2\varepsilon+O(\varepsilon^2)<2k_2\varepsilon,$$

and

$$\left|\frac{\partial^2\Phi_{i+1}}{\partial Z^2}\right|<\left|\frac{\partial^2\Phi_3}{\partial Z^2}\right|+O(\varepsilon)<2k_2,\quad |\Psi_{i+1}(\varphi, y, 0)|<|\Psi_3(\varphi, y, 0)|+O(\varepsilon^4)<2k_2\varepsilon.$$

Hence inequalities (3.7) hold for $j = i + 1$, i.e., they hold for all j as long as D_j is not empty. For $j=[(1/4)k_1^{-1}K^{-1}\varepsilon^{-1}]-1$ we have $|h_j|<M_j=O(\exp(-c_2^{-1}/\varepsilon))$.

3.3. Phase Elimination. The phase φ is eliminated from the system for $|y-y_*|<c_3^{-1}$, in the principal approximation, when Lemma 1 is applied to (3.1). This is established in two steps. We first introduce the new variable

$$\hat{y}=y-\varepsilon B(\varphi, y)/\omega(y),\quad B(\varphi, y)=\int_0^{\varphi}(b(\vartheta, y)-G(y))\,d\vartheta,$$

and then, making the analytic Lyapunov variable change $\hat{z}=\Lambda(\varphi, \hat{y})Z$, 2π-periodic in φ ([13], pp. 86 and 298), we obtain a linearized system for Z with $\hat{y}=\text{const}$, $\dot{\varphi}=\omega(\hat{y})$, and $\varepsilon = 0$, and constant coefficients; the matrix $\Lambda(\varphi, y)$ is in general complex for real φ and y. The equations now become

$$\dot{z}=A(y)z+\varepsilon O(|z|)+O(|z|^2)+O(\exp(-c_2^{-1}/\varepsilon)),$$
$$\dot{y}=\varepsilon G(y)+\varepsilon O(|z|)+O(\varepsilon^2),\quad \dot{\varphi}=\omega(y)+O(|z|)+O(\varepsilon) \tag{3.8}$$

(we drop the hats over the new variables). For $\tau < \tau_*$ the real parts of the eigenvalues of the matrix $A(Y(\tau))$ are negative, and for $\tau = \tau_*$ the matrix has at least one eigenvalue with zero real part.

3.4. A Lyapunov Function. We follow [14] (p. 272) and introduce the function $V(z, \tau) = \int_0^{\infty}|\exp(A(Y(\tau))t_1)z|^2dt_1$. For $\tau\in[\tau_*-c_4^{-1}, \tau_*)$ the function V satisfies the inequalities $\nu_1(\tau)|z|^2<V(z, \tau)<\nu_2(\tau)|z|^2$, where the $\nu_i(\tau)$ are positive and continuous on the half-open interval indicated. By virtue of the system $\dot{z} = A(Y(\tau))z$, $\tau = \text{const}$, the derivative of V is $(-|z|^2)$. Let $\alpha=\exp(-c_2^{-1}/\varepsilon)$.

LEMMA 2. If $\alpha/\varepsilon<|z|<c_1^{-1}$, $\tau_*-c_4^{-1}<\tau<\tau_*$, and $|y-Y(\tau)|<c_5^{-1}$, then $\dot{V}<-|z|^2(1-\mu(\tau)(\varepsilon+|y-Y(\tau)|))$, where $\mu(\tau)$ is a positive continuous function on $[\tau_*-c_4^{-1}, \tau_*)$.

Proof. The function $\dot{V}$ differs from $-|z|^2$ by terms that are $O(\varepsilon|z|^2)$, $O(|y-Y(\tau)||z|^2)$, and $O(\alpha|z|)$. For $|z|>\alpha/\varepsilon$, the last term is $O(\varepsilon|z|^2)$. The proportionality factor increases for $\tau \to \tau_* - 0$, and is majorized by some function $\mu(\tau)$.

3.5. Description of the Motion. Results in [8] imply that the phase point $[x(t), y(t)]$ of the fast-slow system, beginning for $\tau = \tau_0$, $y(t) = y_0$, in the C_1^{-1}-neighborhood of the "state" Ly_0, will for $\tau_0+C_4\varepsilon|\ln\varepsilon|\leqslant\tau\leqslant\tau_*-c_4^{-1}/4=\tau_3$ be in an $O(\varepsilon)$-neighborhood of the "state" $LY(\tau)$. For $\tau>\tau_*-c_4^{-1}/2=\tau_2$, this point can be characterized by the variables (z, φ, y), introduced in Sec. 3.3, and we denote the point by $(z(t), \varphi(t), y(t))$. It follows from Lemma 2 that, if $\tau_2\leqslant\tau\leqslant\tau_3$ and $|z(t)|>\alpha/\varepsilon$, then $\dot{V}<-|z|^2/2<-c_5^{-1}V$. Hence, for $\tau = \tau_3$, the quantity V, and hence $|z|$, are exponentially small: $|z(\tau_3/\varepsilon)|=O(\exp(-c_6^{-1}/\varepsilon))$.

Let $T_*(\varepsilon)$ denote the upper bound of values of $T\in[\tau_3, \tau_*]$ such that $\mu(\tau)<1/\sqrt{\varepsilon}$, $\nu_2(\tau)<1/\varepsilon$, and $\nu_1(\tau)>\varepsilon$ for $\tau_3\leqslant\tau\leqslant T$. Clearly $T_*(\varepsilon)\to\tau_*$ for $\varepsilon \to 0$. Let $T_*^!$ be the upper bound of slow times $\tau'\in[\tau_3, T_*]$ such that $|z(t)|<\varepsilon$ and $|y(t)-Y(\varepsilon t)|<1/3\sqrt{\varepsilon}$ for $\tau_3\leqslant\varepsilon t\leqslant\tau'$. Lemma 2 implies that, when $\tau_3 \leqslant \varepsilon t \leqslant T'_*$ and $|z| > \alpha/\varepsilon$, we have $\dot{V} \leqslant -1/2|z|^2 < 0$. Hence, on the time interval under consideration, $V[z(t), \varepsilon t]$ and so $|z(t)|$ remain exponentially small: $|z(t)| = O(\exp(-c_7^{-1}/\varepsilon))$. The equation for y becomes $\dot{y} = \varepsilon G(y) + O(\varepsilon^2)$. Hence $|y(t) - Y(\varepsilon t)| = O(\varepsilon)$ and the definition of T'_* implies that $T^!_* = T_*$.

For $|z(t)|$ to increase from an exponentially small value for $t = T_*(\varepsilon)/\varepsilon$ to the value c_3, a time at least as large as $2C_2^{-1}/\varepsilon$ is required. Using the inequality $\tau_* - T_*(\varepsilon) < C_2^{-1}$, and returning to the original variables, we conclude that, for $\tau_0 + C_4\varepsilon|\ln\varepsilon| < \varepsilon t < \tau_* + C_2^{-1}$, the phase point is in the $O(\varepsilon)$-neighborhood of the "state" $L_{Y(\varepsilon\tau)}$, and this is the conclusion of Theorem 1.

4. Nonanalytic Systems. We return to the equations in Sec. 1, and consider the general case in which the eigenvalues (multipliers) corresponding to a stability loss for $\tau = \tau_*$, are not multiple, and their real parts (the real parts of the logarithms of the multipliers) pass through 0 with zero velocity ν.

THEOREM 2. If the right sides in (1.1) have $l \geqslant 2$ continuous derivatives with respect to x, y, then the phase point $[x(t), y(t)]$, beginning for $\varepsilon t = \tau_0$, $y(t) = y_0$ in the C_1^{-1}-neighborhood of the "state" L_{y_0} will, for $\varepsilon t \in [\tau_*, \tau_* + \sqrt{2(l-3/2)\varepsilon|\ln\varepsilon|/\nu}]$, be in the $C_3\varepsilon$-neighborhood of the "state" $L_{Y(\varepsilon t)}$. If the right sides are infinitely differentiable, then the phase point will be in the neighborhood indicated for $\varepsilon t \in [\tau_*, \tau_* + \sqrt{M(\varepsilon)\varepsilon|\ln\varepsilon|}]$ with $M(\varepsilon) \to +\infty$ monotonically for $\varepsilon \to 0$.

Outline of Proof. A cycle can be transformed into an equilibrium state of a system with periodic coefficients, as in 3.1. To the resulting system we apply the successive-substitution procedure described in 3.2; however when the system has finite smoothness the procedure ends after $\ell - 1$ steps. The free term in (3.2) is therefore of order ε^ℓ. In the new variables, the motion can be described as follows. The phase point is in the $O(\varepsilon^\ell)$-neighborhood of the stable-equilibrium state $z = 0$, and remains there up to at least the stability-loss time τ_*. Departure from equilibrium for small $\tau - \tau_* > 0$ proceeds not faster than for an unstable linear inhomogeneous system whose eigenvalue increases smoothly:

$$|z(t)| = O(\varepsilon^{l-1/2}\exp(\nu\varepsilon^{-1}(\varepsilon t - \tau_*)^2/2)).$$

For $\varepsilon t \leqslant \tau_* + \sqrt{2(l-3/2)\varepsilon|\ln\varepsilon|/\nu}$, the phase point is in the $O(\varepsilon)$-neighborhood of the state $L_{Y(\tau)}$.

In the infinitely smooth case the successive-substitution procedure can be terminated after an arbitrary number of steps. Hence, for arbitrary N, there is $\varepsilon_1 = \varepsilon_1(N)$ such that, for $0 < \varepsilon \leqslant \varepsilon_1$, the phase point is in the $C_4^{-1}\varepsilon$-neighborhood of the state $L_{Y(\tau)}$ for $\tau_* \leqslant \tau \leqslant \tau_* + \sqrt{2N\varepsilon|\ln\varepsilon|/\nu}$, and $\varepsilon_1(N)$ tends monotonically to 0 for $N \to \infty$. Let $N_1(\varepsilon)$ be the step function inverse to $\varepsilon_1(N)$. Putting $M(\varepsilon) = 2N_1(\varepsilon)/\nu$, we obtain the second conclusion of the theorem.

The following examples show that the bounds in Theorem 2 cannot be improved.

Example 1. Consider the linear inhomogeneous system in the complex form

$$\dot{z} = (\tau + i)z + \varepsilon h(\tau), \quad z = z_1 + iz_2, \quad \tau = \varepsilon t. \tag{4.1}$$

The equilibrium position $z = 0$ of the fast system loses stability for $\tau = 0$. Let h be infinitely differentiable for $\tau \neq 0$, and let its ℓ-th derivative be discontinuous for $\tau = 0$: $h^{(l)}(+0) - h^{(l)}(-0) = a \neq 0$.

Let $z_-(\tau)$ and $z_+(\tau)$ be solutions of (4.1), bounded for $\tau < 0$ and $\tau > 0$, respectively. Let $z_-(\tau) = O(\varepsilon)$ for $\tau < 0$ and let $z_+(\tau) = O(\varepsilon)$ for $\tau > 0$. Solving (4.1), we obtain

$$z_-(\tau) - z_+(\tau) = e^{(\tau^2/2 + i\tau)/\varepsilon}\int_{-\infty}^{\infty} e^{-(\vartheta^2/2 + i\vartheta)/\varepsilon}\, h(\vartheta)\, d\vartheta = e^{(\tau^2/2+i\tau)/\varepsilon}((-i\varepsilon)^{l+1}a + O(\varepsilon^{l+2})).$$

Here the last equality is obtained by $(\ell + 1)$ integrations by parts. For $\tau = \sqrt{2(l+1)\varepsilon|\ln\varepsilon|}$ we have $|z_-(\tau)| = a + O(\varepsilon) \sim 1$, so that the order with respect to ε of the bound in Theorem 2 cannot be improved in the finite-smoothness case.

Example 2. Let

$$h(\tau) = \int_0^{1/2} \frac{e^{q(\rho)\ln\rho}}{(\tau + i\rho)}\, d\rho, \quad q(\rho) > 0, \quad q(\rho) \to +\infty \quad \text{for} \quad \rho \to +0 \tag{4.2}$$

$$z_-(\tau) - z_+(\tau) = e^{(\tau^2/2+i\tau)/\varepsilon} \int\limits_0^{1/2} e^{q(\rho)\ln\rho}\, d\rho \int\limits_{-\infty}^{\infty} \frac{e^{-(\vartheta^2/2+i\vartheta)/\varepsilon}}{\vartheta + i\rho}\, d\vartheta.$$

The interior integral, calculated on the contour in Fig. 2, has its principal part at the pole $\vartheta = -i\rho$, and we obtain

$$z_-(\tau) - z_+(\tau) = 2\pi i e^{(\tau^2/2+i\tau)/\varepsilon} \Big[\int\limits_0^{1/2} e^{q(\rho)\ln\rho - \rho/\varepsilon + \frac{1}{2}\rho^2/\varepsilon}\, d\rho + O(e^{-1/(2\varepsilon)}) \Big]. \tag{4.3}$$

Let $J(\varepsilon)$ denote the integral in (4.3), let $\delta(\varepsilon)$ be the root of the equation $\delta/(q(\delta)\ln\delta) = -\varepsilon$, and let $\beta(\varepsilon) = -\delta(\varepsilon)/(\varepsilon\ln\varepsilon)$. Under our assumptions, $\delta(\varepsilon)$ is uniquely defined and $\beta(\varepsilon) \to +\infty$ for $\varepsilon \to 0$. The inequality

$$J(\varepsilon) > \int\limits_{\delta(\varepsilon)}^{1/2} e^{-2\rho/\varepsilon} d\rho = \frac{\varepsilon}{2} e^{2\beta(\varepsilon)\ln\varepsilon} + O(\varepsilon e^{-1/\varepsilon})$$

holds, and so $|z_-(\tau)| \gg 1$ for $\tau = \sqrt{5\beta(\varepsilon)\varepsilon|\ln\varepsilon|}$. Hence, for the system under consideration, $M(\varepsilon) < 5\beta(\varepsilon)$ in Theorem 2. The function $\beta(\varepsilon)$ is arbitrary, except that it must be such that $\beta(\varepsilon) \to +\infty$ and $\beta(\varepsilon)\varepsilon|\ln\varepsilon| \to 0$ monotonically for $\varepsilon \to 0$; thus, by inverting the preceding formulas, we obtain the corresponding function $q(\rho)$. It is thus clear that the bound in Theorem 2, for the inifinite-differentiability case, cannot be improved by replacing $M(\varepsilon)$, to within a proportionality constant, by a slowly increasing function identical for all systems.

5. Stability-Loss Persistence for a Fixed point. An analog of a fast–slow differential equation system is a fast–slow mapping

$$x \mapsto x + f(x, y, \varepsilon), \quad y \mapsto y + \varepsilon g(x, y, \varepsilon). \tag{5.1}$$

The fast mapping is the mapping x for y = const and $\varepsilon = 0$. Suppose that, for each y, it has a nonsingular fixed point depending continuously on y. Let L_y denote this point in R^{n+m}, and consider the slow system $\dot{y} = \varepsilon G(y)$, where G is the value of g at L_y for $\varepsilon = 0$. Fix a solution of the slow system $y = Y(\tau)$, let $\tau = \varepsilon t \in [\tau_0, \tau_1]$, and let $Y(\tau_0) = y_0$. Suppose that the fixed point $L_{Y(\tau)}$, for some $\tau = \tau_*$, loses its stability but remains a fixed point (this situation was suggested by Yu. A. Kravtsov and A. I. Libkin).

THEOREM 3. Let the fast–slow mapping (5.1) be continued analytically into a neighborhood of the point $L_{Y(\tau_*)}$ not depending on ε, but remaining smooth with respect to ε. If a point (x_0, y_0) is in the C_1^{-1}-neighborhood of L_{y_0}, then the n-th iteration of the fast–slow mapping is in the C_3^{-1}-neighborhood of $L_{Y(\varepsilon n)}$ for $\tau_* \leqslant \varepsilon n \leqslant \tau_* + C_2^{-1}$.

The proof is similar to the proof of Theorem 1.

There is in general no stability-loss persistence for nonanalytic mappings, similar to that in Sec. 4.

The author takes this opportunity to thank V. I. Arnol'd for suggesting this problem and D. V. Anosov for his discussion of the results.

LITERATURE CITED

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KNESER'S THEOREM

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UDC 517.911

In this article we discuss the property of a space of solutions of an ordinary differential equation, established by Kneser ([1], p. 28), using the axiomatic approach to Cauchy's theorem for ordinary differential equations and the corresponding differential inclusions developed in [2] (see also [3, 4]). We also give a generalization of Hartman's theorem ([1], p. 247). We thus clarify the role of the Kneser property and show that the method of decreasing sequences of differential inclusions [3, 4] can be applied, without introducing extra difficulties, in the investigation of whether Kneser's property prevails.

Let U be an arbitrary open subset of R × L, where R is the real line and L is a finite-dimensional Euclidean space. Let $C_s(U)$ be the set of all continuous functions defined on finite intervals and one-point subsets of the real line, with values in L and graphs in U. We write Gr for the mapping of the elements of $C_s(U)$ into the graphs of these elements (these graphs form a compact subset of U), which maps $C_s(U)$ injectively into the space of compact subsets of U. The Vietoris topology, generated in the case under consideration by the Hausdorff metric ([5], p. 223), is passed on by the mapping Gr to the set $C_s(U)$. This is the topology implied when we speak of the topological space $C_s(U)$. We write $\pi(z)$ for the domain of definition of a function z, and we use the notation $Z_M=\{z : z\in Z, \mathrm{Gr}(z)\subseteq M\}$ when $Z\subseteq C_s(U)$ and $M\subseteq U$.

Let R(U) denote the set of subsets of $C_s(U)$, closed with respect to the restriction of their elements to intervals in the domain of definition, and with respect to the composition of common mutual continuation of pairs of functions coinciding on nonempty intersections of their domains of definition; we write $R_{ce}(U)$ for the set of spaces $Z\in R(U)$ satisfying the following conditions: a) for each compact subset K of U, the set Z_K is compact; b) for each point (t, y) of U, there is a function from the set Z, defined on a segment containing t as an interior point, and with the value y at the point t (for more details see [2, 3] or [4]). If the right side of an equation or inclusion satisfies the conditions of Peano's, Carathéodofy's, or Devy's theorem, then the space of solutions is in $R_{ce}(U)$.

If $Z\in R(U)$, we write Z^+ (Z^-) for the set of all continuous mappings $z:[a, b[\to L$ ($z:]a, b] \to L$; the open end can be $+\infty$ or $-\infty$), satisfying the following conditions: if I is a finite interval in the domain of z, then $z|_I\in Z$ and is not an element of the space Z continuing the function z. We write Z^{-+} for the set of continuous mappings $z:]a, b[\to L$ such that $z|_{]a, t]}\in Z^-$ and $z|_{[t, b[}\in Z^+$ for $t\in]a, b[$.

LEMMA 1. Let $Z\in R(U)$ and, for each compact subset K of U, let Z_K be compact. Let $z: [a, b[\to L$ and let $z\in Z^+$. Then, for $t \to b$, the graph of z(t) leaves each compact subset of U, i.e., for each compact $K\subseteq U$, there is $t_0\in [a, b[$ such that $(t, z(t))\in U\setminus K$ for $t > t_0$.

Proof. We first note that there is no subset K of U containing the graph of z, since otherwise the compactness of Z_K implies that the sequence of restrictions of z to the intervals $[a, b - 2^{-n}(b - a)]$ (we cannot have $b = \infty$ because K is compact) contains a subsequence

M. V. Lomonosov Moscow State University. Translated from Differentsial'nye Uravneniya, Vol. 23, No. 12, pp. 2068-2074, December, 1987. Original article submitted January 28, 1986.