Stochastic Partial Differential Equations and Applications II. Edited by G. Da Prato and L. Tubaro. Springer-Verlag, 1989. vi+258 pp., $24.30.

This is Volume 1390 of Lecture Notes in Mathematics. It is the proceedings of the second meeting on the title subject, held in Trento, Italy, February 1–6, 1988. There are 22 papers, including applications to subjects such as filtering theory, biological models, control theory, field theory in physics.

Simulating Computer Systems: Techniques and Tools. By M. H. MacDougall. The MIT Press, 1989. 292 pp., $27.50.

This is a volume in the MIT Press Series in Computer Systems. It is intended for computer and communication system designers who want to analyze the performance of their designs using simulation. It provides an introduction to discrete-event simulation, including model building and output data analysis, and presents a discrete-event simulation language called SMPL. It is divided into two parts, the first providing an introduction to discrete event simulation using SMPL, the second giving an overview of the SMPL environment.

Three Hundred Years of Gravitation. Edited by S. W. Hawking and W. Israel. Cambridge University Press, 1989. xiii+690 pp., $34.50.

This is the first paperback edition (with correction) of the book first published in 1987. It celebrates the 1687 publication of Newton's *Principia* and contains 16 papers by eminent authorities on Newtonian and Einsteinian gravity, with applications to cosmology.

Introduction to Theories of Plasticity, Part I: *Stress-Strain Relations.* By S. J. Jain. Engineering Publications (P.O. Box 302, Blacksburg, Virginia 24060), 1989. 280 pp., $20.00.

This series of lecture notes presents the incremental theory of plasticity from a unified viewpoint advanced by W. Prager and D. Drucker in 1949/50. The notes also present a brief introduction to the deformation theory of plasticity.

Optimization of Stochastic Systems: Topics in Discrete-Time Dynamics, Second Edition. By Masanao Aoki. Academic Press, 1989. x+417 pp., $64.95.

This is a volume in the series Economic Theory, Econometrics, and Mathematical Economies. Whereas the first edition of this book was written mainly for audiences with physical science and engineering backgrounds, this edition has been largely rewritten to make it more accessible also to graduate students and professionals in the social sciences. The scope of this edition is apparent from the chapter headings: 1. Deterministic models and their control problems; 2. Stochastic models; 3. Stochastic control problems; 4. Time series and econometric models: examples; 5. Estimation; 6. Convergence questions; 7. Adaptive control systems and Bayesian optimal control problems; 8. Linear rational expectation models; 9. Approximations in sequential decision processes.

Continued on page 282

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STABILITY OF PLANE WAVE SOLUTIONS OF COMPLEX GINZBURG-LANDAU EQUATIONS

BY

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Abstract. We consider the stability of plane wave solutions of both single and coupled complex Ginzburg-Landau equations and determine stability domains in the space of coefficients of the equations.

1. Introduction. Ginzburg-Landau equations, both real and complex, arise in many problems of physical interest. First derived as long wave amplitude equations by Newell and Whitehead [12] and Segel [20], real Ginzburg-Landau equations describe, for example, convection in binary mixtures near the onset of instability, or flames stabilized on a burner [16]. Complex Ginzburg-Landau (CGL) equations were derived by Stewartson and Stuart [23] who studied plane Poiseuille flows, by Ermentrout [2] who extended the results of Howard and Kopell [3] on $\lambda - \omega$ systems to general reaction diffusion systems, and by others (see also [6, 11] and the references therein).

Actually, CGL equations describe the evolution of the amplitudes of unstable modes for any process exhibiting Hopf bifurcation, for which the continuous band of unstable wave numbers is taken into account. Therefore, the equations have become a self-significant object of study.

Coupled CGL equations govern the amplitudes, on slow spatial and temporal scales, of traveling waves propagating in opposite directions on the faster temporal and spatial scales associated with the original problem from which the Ginzburg-Landau equations were derived. The case when one of the amplitudes equals zero corresponds to a traveling wave solution of the original problem, while the case when the moduli of the two amplitudes are equal corresponds to a standing wave.

The simplest solutions of CGL equations are plane wave solutions and the present paper is devoted to an analysis of their stability. Stability of traveling wave solutions of a single CGL equation, which arises when the most unstable wave number is $k = 0$, was studied by Eckhaus [1], Lange and Newell [8], Stuart and DiPrima [24], and Ermentrout [2] (see also [6] and the references therein) and others. Here we give more complete results for this case (Sec. 2). In addition we consider the case $k \neq 0$, for which, due to reflection symmetry, there are two amplitudes of waves

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propagating in opposite directions, whose evolution as functions of slow temporal and spatial variables is described by coupled CGL equations. The stability of both traveling wave and standing wave solutions of coupled CGL equations is studied in Sec. 3.

We consider the case that the real parts of the coefficients of the diffusive terms are positive, and the real parts of certain nonlinear terms are negative; otherwise, plane wave solutions are necessarily unstable. The equations are scaled so that these terms are $+1$ and -1 respectively. Interesting results for the case that plane wave solutions are unstable, were obtained by Schöpf and Kramer [19], Sirovich et al [13, 14, 18, 21, 22], Keefe [5], Kuramoto and Koga [7], Moon et al [10], Nozaki and Bekki [15], and others.

2. Stability of traveling wave solutions of a single CGL equation. We consider the scaled complex Ginzburg-Landau equation

$$\frac{\partial R}{\partial t} = R + (1+ia)\frac{\partial^2 R}{\partial x^2} + (-1+ib)R|R|^2, \tag{2.1}$$

where a and b are real constants.

We seek solutions in the form of traveling waves

$$R = \rho e^{i(\omega t + kx)}, \tag{2.2}$$

where $\rho \geq 0$, and ω and k are real constants. In addition to the trivial solution $\rho = 0$, we obtain the one-parameter family of solutions

$$\rho^2 = 1 - k^2, \qquad \omega = b - (a+b)k^2. \tag{2.3}$$

To study the stability of the traveling wave solutions we represent R as

$$R = (\rho + r)e^{i(\omega t + kx)}. \tag{2.4}$$

Substituting (2.4) into (2.1) and linearizing we obtain

$$\frac{\partial r}{\partial t} = (1+ia)\left(2ik\frac{\partial r}{\partial x} + \frac{\partial^2 r}{\partial x^2}\right) + (-1+ib)\rho^2(r+\bar{r}). \tag{2.5}$$

We set $r = u + iv$ and seek normal mode solutions of the form

$$u = e^{\Omega t + i\kappa x}\tilde{u}, \qquad v = e^{\Omega t + i\kappa x}\tilde{v}, \tag{2.6}$$

where $\tilde{u}$ and $\tilde{v}$ are constants. Substituting (2.6) into (2.5) we find that nontrivial solutions exist for two values of Ω determined by the dispersion relation

$$\Omega^2 - 2p\Omega + q = 0, \tag{2.7}$$

where

$$p_r = -\kappa^2 - \rho^2, \qquad p_i = -2\kappa k a, \tag{2.8}$$

$$q_r = \kappa^4(1+a^2) - 4\kappa^2 k^2(1+a^2) - 2\rho^2\kappa^2(-1+ab), \tag{2.9}$$

$$q_i = 4\kappa k\rho^2(a+b). \tag{2.10}$$

Here the subscripts r and i denote the real and imaginary parts of the indicated quantity. In order for the plane wave solution to be stable we must have $\Re(\Omega_j) < 0$ $(j = 1, 2)$, which is equivalent to

$$4p_r^2 q_r + 4p_r p_i q_i > q_i^2. \tag{2.11}$$

Substituting (2.8)–(2.10) into (2.11) we obtain

$$a_1\kappa^8 + a_2\kappa^6 + a_3\kappa^4 + a_4\kappa^2 > 0, \tag{2.12}$$

where

$$a_1 = 1 + a^2,$$
$$a_2 = 2\rho^2(1+a^2) - 4k^2(1+a^2) - 2\rho^2(-1+ab),$$
$$a_3 = \rho^4(1+a^2-4(-1+ab)) + 8\rho^2k^2(-1+ab),$$
$$a_4 = -2\rho^4(\rho^2(-1+ab) + 2k^2(1+b^2)).$$

The traveling wave solution (2.2)–(2.3) will be stable if and only if the inequality (2.12) is satisfied for all values of κ. We will derive necessary and sufficient conditions on the coefficients a and b as well as the wave number k for the inequality to be satisfied. To do so we introduce the parameter $\alpha \geq 0$ defined by $k^2 = \alpha\rho^2$. Then taking (2.3) into account, we have

$$k^2 = \frac{\alpha}{1+\alpha}, \qquad \rho^2 = \frac{1}{1+\alpha}. \tag{2.13}$$

Introducing

$$\xi = \frac{-1+ab}{1+a^2}, \quad \eta = \frac{1+b^2}{1+a^2}, \quad y = \frac{\kappa^2}{1+\alpha}, \tag{2.14}$$

we divide (2.12) by κ^2 and rewrite inequality (2.12) as

$$f(y) = y^3 + 2y^2(1-2\alpha-\xi) + y(1-4\xi+8\alpha\xi) - 2\xi - 4\alpha\eta > 0, \tag{2.15}$$

which must be satisfied for all positive y in order that the traveling wave solution (2.2)–(2.3) be stable.

An obvious necessary condition is

$$\xi + 2\alpha\eta < 0,$$

which can be written in terms of the original parameters a and b as

$$2\alpha(1+b^2) + ab < 1. \tag{2.16}$$

If $\alpha = 0$, which corresponds to $k = 0$, (2.16) reduces to

$$ab < 1,$$

which describes the stability region for the Stokes solution, which is a spatially uniform, time periodic solution of (2.1), as discussed, e.g., in [21].

In regions of the (b, a) plane determined by (2.16), traveling wave solutions are stable to long-wave (LW) $(\kappa \ll 1)$ perturbations. The regions are shown in Fig. 1 for different values of α (see pp. 268 and 269).

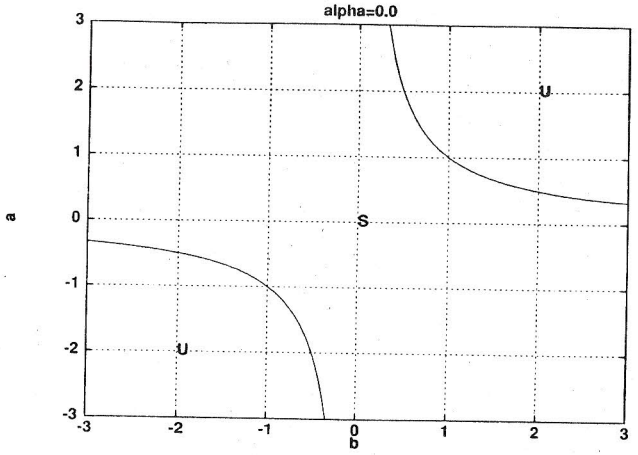

FIG. 1(a). Region of LW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0$.

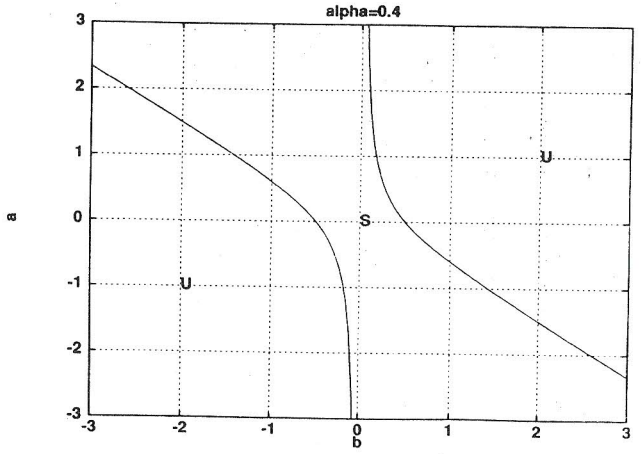

FIG. 1(b). Region of LW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0.4$.

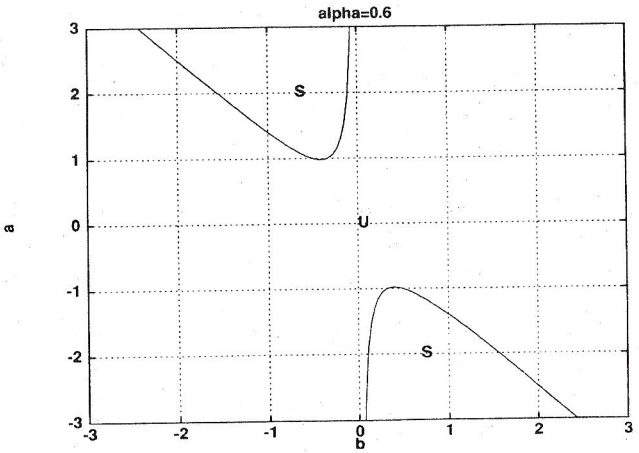

FIG. 1(c). Region of LW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0.6$.

It is clear that condition (2.16) is not sufficient for (2.15) to be satisfied. We now give a brief derivation of sufficient conditions. We consider the critical case which corresponds to the stability boundary in the space of parameters, when for some value of y, say y_*, $f(y)$ has a minimum that is tangent to the axis $f = 0$, i.e.,

$$f(y_*) = 0, \qquad f'(y_*) = 0,$$

assuming that (2.16) is satisfied. Solving these two equations simultaneously yields both the value of y_* as a function of a, b, and α and the relation between a and b at the stability boundary. Actually it is more convenient to determine a and b (equivalently, ξ and η) as functions of y_* to obtain (we omit the subscript)

$$\xi(y) = \frac{3y^2 + 4y(1-2\alpha) + 1}{4(y + 1 - 2\alpha)}, \tag{2.17}$$

$$\eta(y) = \frac{1}{4\alpha}(y^3 + 2y^2(1 - 2\alpha - \xi(y)) + y(1 - 4\xi(y)(1 - 2\alpha)) - 2\xi(y)). \tag{2.18}$$

Taking into account (2.14) we obtain

$$b(y) = \pm\sqrt{\frac{(\xi(y) + \eta(y))^2}{\eta(y) - \xi^2(y)}}, \tag{2.19}$$

$$a(y) = \frac{1}{b(y)}\left(\frac{\xi(y)}{\eta(y)}(1 + b^2(y)) + 1\right). \tag{2.20}$$

Formulas (2.17)–(2.20) provide a parametric representation of the stability boundaries with respect to perturbations with finite wave (FW) numbers in the (b, a)

plane. The range of variation of the parameter y in these formulas is determined by the conditions

$$\xi(y) + 2\alpha\eta(y) \le 0, \tag{2.21}$$

$$\eta(y) > \xi^2(y). \tag{2.22}$$

The former condition corresponds to $f(0) \ge 0$, and the latter condition to the requirement that b is real. It can be shown that the conditions (2.21), (2.22) cannot be satisfied when $\alpha < \frac{2}{3}$ or $\alpha > 1$. However, for all points (b, a) lying inside the stability regions S (see Fig. 1), if $\alpha < \frac{2}{3}$ $(\alpha > 1)$, the traveling wave solution (2.2) is stable (unstable) with respect to FW perturbations. For $\frac{2}{3} < \alpha < 1$, there exists a range of y satisfying (2.21)–(2.22). Therefore, there exist FW stability boundaries inside the LW stability region, that is, S is subdivided into two subregions inside which the traveling wave solutions are respectively stable and unstable to FW perturbations. These are exhibited in Fig. 3 at the end of this paper. This range of variation of y lies in the interval $(0, 1)$ and is given by (see Fig. 2)

$$0 < y < y^*(\alpha) \qquad \left(\frac{2}{3} < \alpha < \frac{3}{4}\right),$$

$$4\alpha - 3 < y < y^*(\alpha) \qquad \left(\frac{3}{4} < \alpha < 1\right),$$

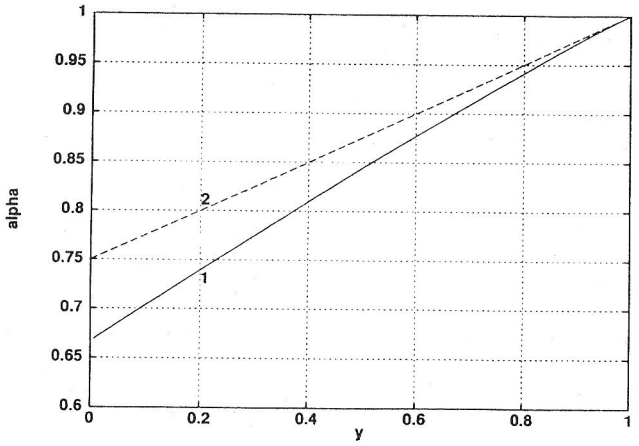

FIG. 2. Ranges of variation of y in the parametric dependence (2.17)–(2.20). For a fixed $\alpha(\frac{2}{3} < \alpha < \frac{3}{4})$, y varies between zero and y^* (see (2.23)), and between $4\alpha - 3$ and y^* for $\frac{3}{4} < \alpha < 1$.

where $y^*(\alpha)$ is the inverse function of

$$\alpha(y) = \frac{1}{2} + \frac{4y^3 + y^2 + 2y + 1}{11y^2 + 2y + 3 + \sqrt{(1-y)^3(7y+9)}}. \tag{2.23}$$

Stability regions in the (b, a) plane are shown in Fig. 3 for different values of α. Points (b, a) lying in the regions S, U, and U1 correspond to traveling wave solutions that are stable, and unstable to LW and FW perturbations, respectively.

Let us consider the trajectory in the (b, a) plane, of the point of intersection of the LW and FW stability boundaries as α varies from $\frac{2}{3}$ to 1. Substituting $y = 0$ for $\frac{2}{3} < \alpha < \frac{3}{4}$ and $y = 4\alpha - 3$ for $\frac{3}{4} < \alpha < 1$ into (2.19)–(2.22), we obtain

$$a = \frac{-2b(b^2-1) + \sqrt{4b^2(b^2-1)^2 + (3b^2-1)(5b^2+1)}}{3b^2 - 1} \qquad \left(b^2 > \frac{1}{3}\right),$$

$$a = -\frac{1}{b} \qquad \left(b^2 < \frac{1}{3}\right).$$

The curves are shown with dashed lines in Fig. 4 at the end of this paper. The solid line in this figure is the curve

$$a = \frac{1}{b}.$$

These curves divide the (b, a) plane into 5 regions. The stability results for traveling wave solutions of a single Ginzburg-Landau equation are as follows. For each pair of coefficients b, a lying in the regions U, traveling waves with any wave number k are unstable. If the coefficients b, a lie in the region S1, there is a critical wave number k_{cr},

$$k_{cr}^2 = \frac{1 - ab}{3 + 2b^2 - ab},$$

such that traveling waves with $k < (>) k_{cr}$ are stable (unstable). The loss of stability when $k = k_{cr}$ is due to LW perturbations. If the coefficients b, a lie in one of the regions S2, then there exists a second critical value of k, say, $k_* = k_*(a, b)$ such that the traveling wave solutions are stable when $k < k_*$ and unstable otherwise. The quantity k_*^2 varies from $\frac{2}{5}$ to $\frac{1}{2}$ for b, a lying in S2. The loss of stability when $k = k_*$ is due to FW perturbations. The value $k_*(a, b)$ can be found explicitly, though we do not present it because the expression is awkward. To find $k_*(a, b)$ we simply solve (2.17) for α, substitute the result into (2.18), solve the resulting equation for y, and thus find α as a function of a and b.

We observe, in particular, that traveling wave solutions cannot be stable for $k^2 > \frac{1}{2}$.

3. Stability of plane wave solutions of coupled complex Ginzburg-Landau equations. Let us now consider the system of coupled complex Ginzburg-Landau equations of

the form

$$\frac{\partial R}{\partial t} = R + (1+ia)\frac{\partial^2 R}{\partial x^2} + (-1+ib)R|R|^2 + cR|S|^2, \tag{3.1}$$

$$\frac{\partial S}{\partial t} = S + (1+ia)\frac{\partial^2 S}{\partial x^2} + (-1+ib)S|S|^2 + cS|R|^2. \tag{3.2}$$

Thus, we consider the case when the group velocity is so small that it is neglected at this order and only enters at higher order in the perturbation analysis which, at this order, yields (3.1)–(3.2). We note that if the group velocity is not small, the amplitude equations are nonlocal Ginzburg-Landau equations involving averaged terms [4, 9, 17]. We seek solutions of the form

$$R = \rho_1 e^{i(\omega_1 t + k_1 x)}, \qquad S = \rho_2 e^{i(\omega_2 t + k_2 x)},$$

where ρ_j, k_j, and ω_j $(j = 1, 2)$ are constants. In addition to the trivial solution $\rho_1 = \rho_2 = 0$, we obtain the following nontrivial solutions:

$$\rho_1^2 = 1 - k_1^2, \quad \rho_2 = 0, \quad \omega_1 = b - (a+b)k_1^2, \tag{3.3}$$

$$\rho_1 = 0, \quad \rho_2^2 = 1 - k_2^2, \quad \omega_2 = b - (a+b)k_2^2, \tag{3.4}$$

and

$$\rho_1^2 = \frac{1 - k_1^2 + c_r(1-k_2^2)}{1 - c_r^2}, \qquad \rho_2^2 = \frac{1 - k_2^2 + c_r(1-k_1^2)}{1 - c_r^2},$$
$$\omega_1 = -ak_1^2 + b\rho_1^2 + c_i\rho_2^2, \qquad \omega_2 = -ak_2^2 + b\rho_2^2 + c_i\rho_1^2. \tag{3.5}$$

The single mode solutions (3.3) and (3.4) correspond to traveling wave solutions of the original problem, while the mixed mode solution (3.5) corresponds to a quasiperiodic wave in the original problem. If $k_1 = k_2$, the solution (3.5) corresponds to a standing wave.

We consider infinitesimal perturbations r_1 and r_2 given by

$$R = (\rho_1 + r_1)e^{i(\omega_1 t + k_1 x)}, \qquad S = (\rho_2 + r_2)e^{i(\omega_2 t + k_2 x)}.$$

Substituting into (3.1)–(3.2) and linearizing we obtain

$$\frac{\partial r_1}{\partial t} = a_{11}r_1 + (-1+ib)\rho_1^2\bar{r}_1 + \rho_1\rho_2 c(r_2 + \bar{r}_2) + (1+ia)\left(2ik_1\frac{\partial r_1}{\partial x} + \frac{\partial^2 r_1}{\partial x^2}\right), \tag{3.6}$$

$$\frac{\partial r_2}{\partial t} = a_{21}r_2 + (-1+ib)\rho_2^2\bar{r}_2 + \rho_1\rho_2 c(r_1 + \bar{r}_1) + (1+ia)\left(2ik_2\frac{\partial r_2}{\partial x} + \frac{\partial^2 r_2}{\partial x^2}\right), \tag{3.7}$$

where the coefficients a_{11} and a_{21} are

$$a_{11} = 1 + i\omega_1 - (1+ia)k_1^2 + 2(-1+ib)\rho_1^2 + c\rho_2^2,$$
$$a_{21} = 1 + i\omega_2 - (1+ia)k_2^2 + c\rho_1^2 + 2(-1+ib)\rho_2^2.$$

We first consider the stability of the traveling wave solutions. We treat the case (3.3) only, since the case (3.4) can be considered in a similar manner. We set $r_j =$

$u_j + iv_j$ $(j = 1, 2)$ and look for normal mode solutions of the form

$$\begin{pmatrix} u_j \\ v_j \end{pmatrix} = e^{\Omega t + i\kappa x}\begin{pmatrix} \tilde{u}_j \\ \tilde{v}_j \end{pmatrix}, \tag{3.8}$$

where $\tilde{u}_j$ and $\tilde{v}_j$ are constants. Substituting (3.8) into (3.6)–(3.7), we find that nontrivial solutions exist for four values of Ω. Since for the single mode solutions the two equations (3.6)–(3.7) decouple, two of the values of Ω satisfy exactly the same equation (2.7) as for a single Ginzburg-Landau equation, while the other two satisfy

$$\Omega^2 - 2\hat{p}\Omega + \hat{q} = 0, \tag{3.9}$$

where

$$\hat{p} = 1 + c_r\rho_1^2 - \kappa^2, \qquad \hat{q} = \hat{p}^2 + (a\kappa^2 - c_i\rho_1^2)^2.$$

Therefore, in order for the traveling wave solution to be stable, we must have exactly the same conditions on the coefficients a and b and on the wave number k_1 $(k_1^2 = \alpha/(1+\alpha))$ as for the single Ginzburg-Landau equation and, in addition,

$$c_r < 0, \qquad k_1^2 < 1 + \frac{1}{c_r}.$$

These restrictions follow from the condition that the solutions of (3.9) must have negative real parts. We observe, in particular, that for the traveling wave solutions to be stable, it is necessary that

$$c_r < -1. \tag{3.10}$$

We now proceed to study the stability of the standing wave solutions $(k_1 = k_2 \equiv k$ in (3.5)). We observe from (3.5) that in this case $\rho_1 = \rho_2 \equiv \rho$ with

$$\rho^2 = \frac{1 - k^2}{1 - c_r}.$$

We find that nontrivial solutions of (3.6)–(3.7) of the form (3.8) exist for four values of Ω determined by

$$\Omega_\pm^2 - 2p^\pm\Omega_\pm + q^\pm = 0,$$

where

$$p_r^\pm = -\kappa^2 + (-1 \pm c_r)\rho^2, \qquad p_i^\pm = -2\kappa k a,$$
$$q_r = \kappa^4(1+a^2) - 4\kappa^2k^2(1+a^2) - 2\rho^2\kappa^2((-1 \pm c_r) + a(b \pm c_i)),$$
$$q_i = 4\kappa k\rho^2(-a(-1 \pm c_r) + (b \pm c_i)).$$

In order for the standing wave to be stable, we must have, for all κ, that

$$p_r^\pm < 0, \tag{3.11}$$

$$4(p_r^\pm)^2 q_r^\pm + 4p_r^\pm p_i^\pm q_i^\pm > (q_i^\pm)^2. \tag{3.12}$$

The conditions (3.11) imply that

$$-1 < c_r < 1, \tag{3.13}$$

while the conditions (3.12) can be reduced to

$$y_\pm^3 + 2y_\pm^2(1 - 2\alpha - \xi_\pm) + y_\pm(1 - 4\xi_\pm + 8\alpha\xi_\pm) - 2\xi_\pm - 4\alpha\eta_\pm > 0, \tag{3.14}$$

where

$$\xi_\pm = \frac{-1 + a((b \pm c_i)/(1 \mp c_r))}{1 + a^2}, \tag{3.15}$$

$$\eta_\pm = \frac{1 + (b \pm c_i)^2/(1 \mp c_r)^2}{1 + a^2}, \tag{3.16}$$

$$y_\pm = \frac{\kappa^2}{\rho^2(1 \mp c_r)}, \tag{3.17}$$

$$\alpha = \frac{k^2}{1 - k^2}. \tag{3.18}$$

We observe that the left-hand side of (3.14) has exactly the same cubic form as in (2.14), where the coefficient b in (2.16) is replaced by $(b \pm c_i)/(1 \mp c_r)$. Therefore, all the stability results for a traveling wave solution of a single Ginzburg-Landau equation, that we found in terms of the parameters a and b, are valid for the stability of the standing wave solution of the coupled complex Ginzburg-Landau equations with the replacement of b by $(b \pm c_i)/(1 \mp c_r)$, and with the additional restriction (3.13).

Comparing the necessary conditions (3.10) and (3.13) for traveling wave and standing wave solutions, respectively, to be stable, we observe that they cannot be stable simultaneously.

4. Summary. We considered a single Ginzburg-Landau equation

$$\frac{\partial R}{\partial t} = R + (1 + ia)\frac{\partial^2 R}{\partial x^2} + (-1 + ib)R|R|^2$$

and studied the stability of traveling wave solutions

$$R = \rho e^{i(\omega t + kx)},$$
$$\rho^2 = 1 - k^2, \qquad \omega = b - (a + b)k^2.$$

The results are shown in Fig. 4. The dashed lines are given by

$$a = \frac{-2b(b^2 - 1) + \sqrt{4b^2(b^2 - 1)^2 + (3b^2 - 1)(5b^2 + 1)}}{3b^2 - 1} \qquad \left(b^2 > \frac{1}{3}\right),$$
$$a = -\frac{1}{b} \qquad \left(b^2 < \frac{1}{3}\right),$$

and the solid line is the curve

$$a = \frac{1}{b}.$$

These curves divide the (b, a) plane into 5 regions. For each point b, a lying in a region U, traveling waves with any wave number k are unstable. In this region,

chaotic behavior was observed numerically (see e.g. [6]). If the coefficients b, a lie in the region S1, there is a critical wave number k_{cr},

$$k^2 < k_{cr}^2 \equiv \frac{1 - ab}{3 + 2b^2 - ab},$$

such that traveling waves with $k < (>)k_{cr}$ are stable (unstable). The loss of stability when $k = k_{cr}$ is due to LW perturbations ($\kappa \ll 1$). If the coefficients b, a lie in one of the regions S2, then there exists a critical value of k, say, $k_* = k_*(a, b)$ such that the traveling wave solutions are stable when $k < k_*$, and unstable otherwise. The quantity k_*^2 varies from $\frac{2}{5}$ to $\frac{1}{2}$ for b, a lying in S2. The loss of stability when $k = k_*$ is due to FW perturbations. The value $k_*(a, b)$ can be found explicitly, though we do not present the formula because it is awkward.

We observe, in particular, that traveling wave solutions cannot be stable for $k^2 > \frac{1}{2}$.

We also considered the stability of both traveling wave solutions

$$R = \rho e^{i(\omega t + kx)}, \qquad S = 0,$$
$$\rho^2 = 1 - k^2, \qquad \omega = b - (a + b)k^2,$$

and standing wave solutions

$$R = \rho e^{i(\omega t + kx)}, \qquad S = \rho e^{i(\omega t + kx)},$$
$$\rho^2 = \frac{1 - k^2}{1 - c_r}, \qquad \omega = -ak^2 + b\rho^2 + c_i\rho^2,$$

of coupled complex Ginzburg-Landau equations

$$\frac{\partial R}{\partial t} = R + (1 + ia)\frac{\partial^2 R}{\partial x^2} + (-1 + ib)R|R|^2 + cR|S|^2,$$
$$\frac{\partial S}{\partial t} = S + (1 + ia)\frac{\partial^2 S}{\partial x^2} + (-1 + ib)S|S|^2 + cS|R|^2.$$

The results on the stability of traveling wave solutions are the same as for the single Ginzburg-Landau equation, with the additional restriction

$$c_r < -1, \qquad k^2 < 1 + \frac{1}{c_r}.$$

Stability results for the traveling wave solutions of the single Ginzburg-Landau equation, that we found in terms of the parameters a and b, are also valid for the stability of the standing wave solution of the coupled complex Ginzburg-Landau equations, with the replacement of b by $(b \pm c_i)/(1 \mp c_r)$ and the additional restriction

$$-1 < c_r < 1.$$

It is clear from the above conditions that traveling wave and standing wave solutions cannot be stable simultaneously.

Though we did not explicitly study the stability of quasiperiodic waves, it is qualitatively clear that the results should be similar to those for the standing wave solutions.

That is, the parameter space can be divided into subregions in which the solutions are stable and those in which they are unstable. The existence of regions of stability, at least for quasiperiodic waves corresponding to k_1 close to k_2, follows from that for standing wave solutions (for which $k_1 = k_2$).

The only existing stability results for plane wave solutions of complex Ginzburg-Landau equations appear to be those for the single equation. Our results for scalar equations extend those in [1, 8, 24]. In [6] the stability of traveling wave solutions with respect to long-wavelength perturbations, that is small κ, is discussed. In fact, however, caution must be exercised in extrapolating the result for stability to LW perturbations to larger values of κ. While the stability boundaries in the two cases agree in parts of the (b, a) plane, they can be shown to diverge from one another elsewhere. This is the case, e.g., when the leading term in (2.12) alone does not suffice to determine the stability boundary. In other words, when $\kappa \to 0$ the stability boundaries for FW perturbations converge to those for LW perturbations, but the limit is nonuniform with respect to a and b. The result obtained in [6] for long-wave stability corresponds to our condition (2.16). As we saw above, this condition is only necessary, but not sufficient to insure stability. Thus, for arbitrary κ, an additional stability boundary, not previously obtained, is determined by equations (2.17)–(2.20).

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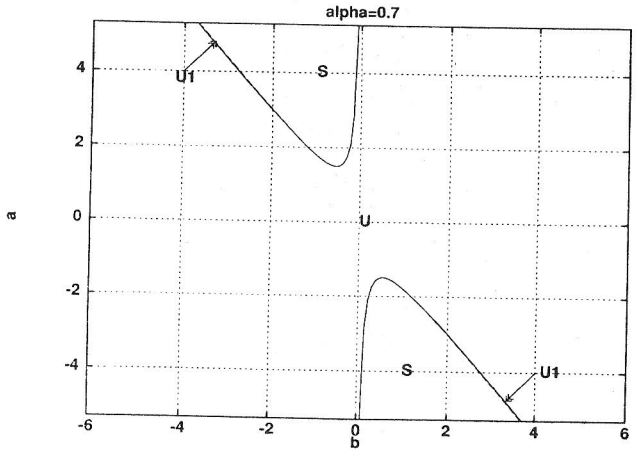

FIG. 3(a). Region of FW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0.7$.

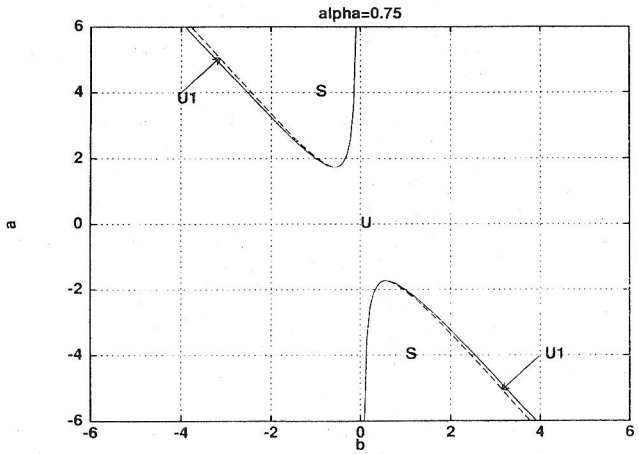

FIG. 3(b). Region of FW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0.75$.

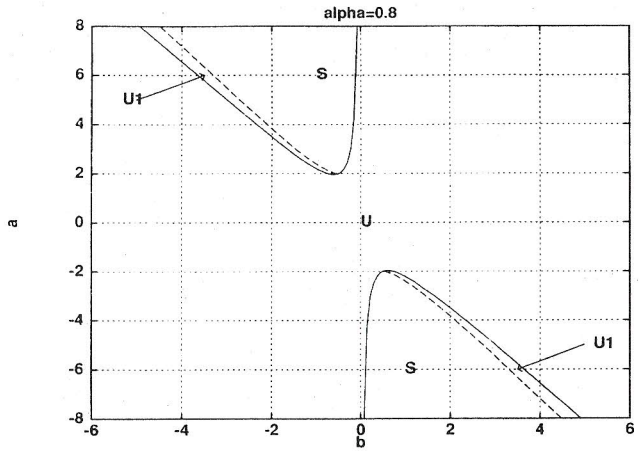

FIG. 3(c). Region of FW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0.8$.

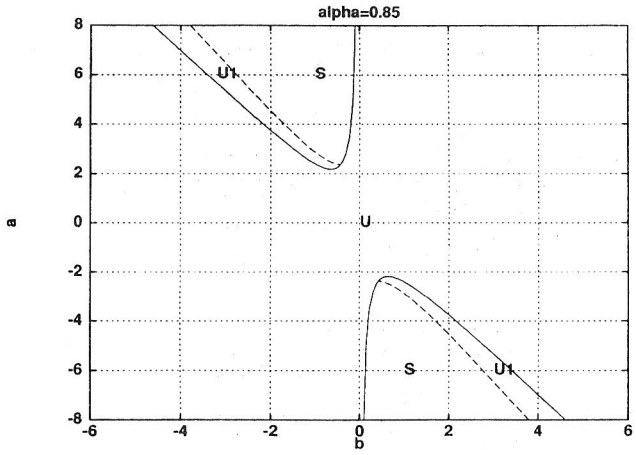

FIG. 3(d). Region of FW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0.85$.

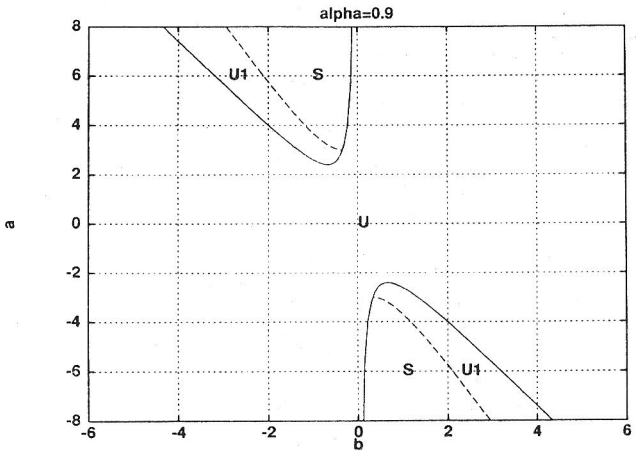

FIG. 3(e). Region of FW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0.9$.

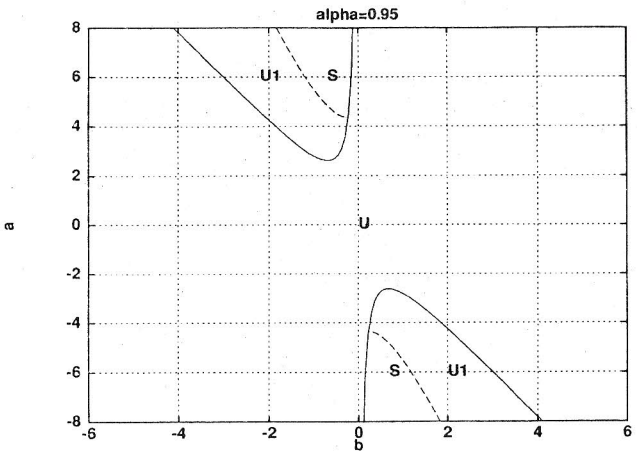

FIG. 3(f). Region of FW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0.95$.

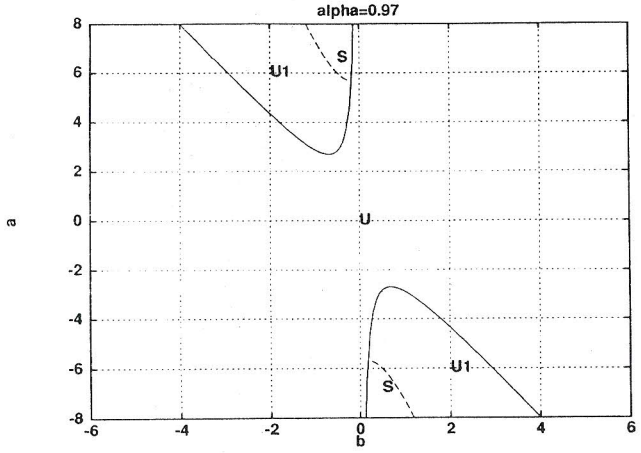

FIG. 3(g). Region of FW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0.97$.

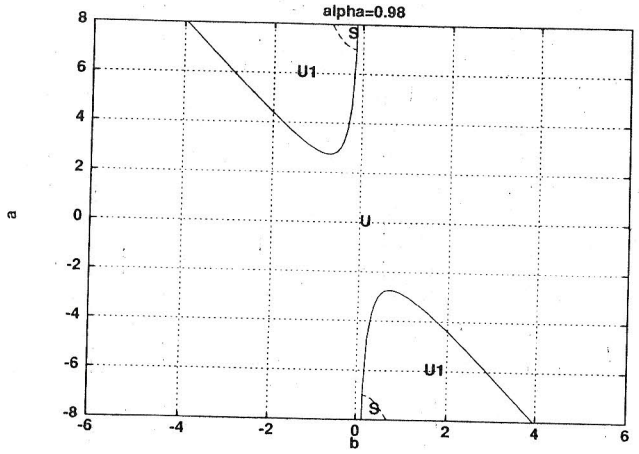

FIG. 3(h). Region of FW stability in the (b, a) plane for a single complex Ginzburg-Landau equation; $\alpha = 0.98$.

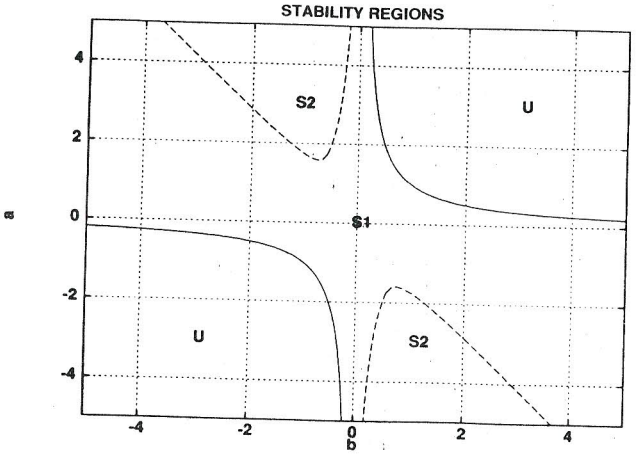

FIG. 4. Regions of stability in the (b, a) plane for a single complex Ginzburg-Landau equation.

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