Exam Applied Analysis

Wednesday 20 December, 10:00-13:00u

- Write your name, studentnumber and major on the first sheet of paper.
- When you use a theorem explain specifically why it applies.

Good luck!

1.) Consider the following system

$$\begin{aligned} \dot{x} &= x^3 - x - 3y \\ \dot{y} &= x^3 - 4y \end{aligned}$$

where the dot means differentiation with respect to t.

(a) Determine the fixed points and characterise these.

(b) Determine the unstable and stable subspaces, E^u and E^s, of all the saddle-points and nodes.

(c) Determine the nullclines and sketch these in the (x, y)-plane.

(d) Show that the line $y = x$ is invariant.

(e) Introduce $z = x - y$. Give the differential equation that z must satisfy and solve this equation for z. What does this imply for $|x(t) - y(t)|$ as $t \to \infty$? What does this imply for the solutions $(x(t), y(t))$ as $t \to \infty$?

(f) Complete the sketch of the phase-plane including stable and unstable manifolds and the flow close to the fixed points. Also, use the behaviour for $t \to \infty$ as found in e.

(g) Determine all possible heteroclinic and homoclinic orbits. Sketch them in the phase-plane in colour and give which fixed points they connect.

2.) Consider the following system

$$\begin{aligned} \dot{x} &= x - y - 2x(x^2 + y^2) \\ \dot{y} &= x + y + xy - 2y(x^2 + y^2). \end{aligned}$$

(a) Characterise the fixed point at the origin.

(b) Rewrite the system in polar coordinates. Prove that the system has a stable limit cycle.

!! Continued on the other side !!

3.) Consider

$$\dot{x} = \mu x - \frac{x}{1+x}$$

where $\mu \in \mathbb{R}$.

(a) Determine all the critical points and determine for which μ they are stable resp. unstable.

(b) For which points (x_c, μ_c) does a bifurcation take place ? Sketch the bifurcation diagram and state the type of bifurcation.

4.) Consider the following system

$$\begin{aligned} \dot{x} &= (ax + 2y)(z + 1) \\ \dot{y} &= (-x + ay)(z + 1) \\ \dot{z} &= -z^3 \end{aligned}$$

where $a \in \mathbb{R}$ is a parameter. The origin is the only fixed point for this system.

(a) Determine the eigenvalues corresponding to the linearisation of the system around $(0, 0, 0)$. For which values of a is the origin unstable ? Determine the value a_c such that there is still a possibility that the origin is stable for $a \leq a_c$.

Note that $(0, 0, 0)$ is not hyperbolic for $a \leq a_c$. We now study the stability of the origin in this case.

(b) Show that the origin is stable for $a = a_c$.

(c) Show that $(0, 0, 0)$ is asymptotically stable for $a < a_c$.
Hint: Take as a Lyapunov function $L = c_1 x^2 + c_2 y^2 + c_3 z^2$.

(d) Take $a < a_c$. Determine the region A for which it holds that all solutions that start in A tend to the origin.