Non-Hausdorff manifolds

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Abstract

A topological manifold is a second countable Hausdorff space that is locally homeomorphic to Euclidean space. We are interested in what happens when we drop the 'Hausdorff' condition. This motivates the following definition:

Definition 0.1. A *pre-manifold* of dimension n is a second countable topological space which admits a cover by open sets U_i such that for each i there exists a open subset $S_i \subset \mathbb{R}^n$ and a homeomorphism $U_i \to S_i$.

A simple example is given by the 'real line with doubled origin' - this is obtained by taking two copies of the real line, and glueing them together along the complement of the origin. Pre-manifolds arise naturally in several situations, such as the study of families of manifolds and vector bundles.

In the next two sections we outline some questions concerning pre-manifolds.

1 A measure of the failure of Hausdorffness

One can show that a pre-manifold M is a manifold if and only if the *diagonal*

$$\begin{aligned} \Delta : M &\to M \times M \\ m &\mapsto (m, m) \end{aligned}$$

is a closed map. Using this, we can define the 'defect of Hausdorffness' $\delta(M)$ of a pre-manifold M by $\delta(M) \stackrel{\text{def}}{=} \overline{\Delta(M)} \setminus \Delta(M)$, i.e. the complement of the diagonal in its closure. Hence M is Hausdorff if and only if its defect is empty.

What can we say about the structure of $\delta(M)$? For example, is it true that δM is constructible (or locally closed) whenever M is compact?

2 A question on compact pre-manifolds

A topological space T is *compact* if every open cover of T admits a finite subcover (we do not require it to be Hausdorff!).

Question 2.1. *Let M be a compact pre-manifold. Is it true that there exists a finite set of compact manifolds C_i and open immersions $C_i \to M$ such that the images of the C_i cover M?*