An *ultrafilter* on a set S is a collection $\mathcal{F} \subseteq 2^S$ of subsets of S satisfying the following three properties: For each $A \subseteq S$, exactly one of A and $S \setminus A$ belongs to $\mathcal{F}$; if $A \subseteq B \subseteq S$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$; and if A and B are elements of $\mathcal{F}$, then $A \cap B \subseteq S$ is also an element of $\mathcal{F}$. Equivalently, ultrafilters are the subsets of 2^S which can occur as the inverse image of $\{1\}$ under a Boolean algebra homomorphism $2^S \to 2 = \{0, 1\}$. There is such a homomorphism for each element s of S, giving rise to the *principal ultrafilters* $\mathcal{F}_s \subseteq 2^S$ defined by $A \in \mathcal{F}_s$ iff $s \in A$. But if S is infinite, then Zorn's lemma implies that there are many nonprincipal ultrafilters. The set of all ultrafilters on S is denoted $\mathcal{U}(S)$.

If S is equipped with a topology, then there is a natural relation between $\mathcal{U}(S)$ and S, called *convergence*. We say that an ultrafilter $\mathcal{F} \in \mathcal{U}(S)$ *converges* to $s \in S$, written $\mathcal{F} \searrow s$, if each open neighborhood of s belongs to $\mathcal{F}$, i.e.

$$\mathcal{F} \searrow s \quad \text{iff} \quad \forall U \ni s : \ (U \text{ open} \Rightarrow U \in \mathcal{F}).$$

For example, the principle ultrafilter $\mathcal{F}_s$ converges to s, because each open neighborhood of s in particular contains s, so belongs to $\mathcal{F}_s$. From the convergence relation, the topology on S can be recovered: an open set is one that is an element of every ultrafilter that converges to one of its points:

$$U \text{ open} \quad \text{iff} \quad \forall s \in U : (\mathcal{F} \searrow s \Rightarrow U \in \mathcal{F}).$$

Therefore, every topological property can in principle be described using the convergence relation. For example:

- S is Hausdorff if and only if each ultrafilter on S converges to at most one point of S.
- S is compact if and only if each ultrafilter on S converges to at least one point of S. (In particular, the convergence relation is a function $\mathcal{U}(S) \to S$ if and only if S is compact Hausdorff.)
- If f is a function from one topological space S to another T, then f is continuous if and only if f preserves the convergence relation, i.e. if $\mathcal{F} \searrow s$, then $f(\mathcal{F}) \searrow f(s)$, where the pushforward ultrafilter $f(\mathcal{F})$ on T is defined by $A \in f(\mathcal{F})$ iff $f^{-1}(A) \in \mathcal{F}$.

Students undertaking this project will have the opportunity to learn about monads, algebras, and 2-categories, and will have the chance to research new correspondences between statements about a topological space and statements about its convergence relation. For more details, talk to Owen Biesel.