SAGE - ALGEBRAIC NUMBER THEORY, NOVEMBER 23, 2010

1. Getting started

First go to the website `https://sage.math.leidenuniv.nl/register` to make an account for Sage Notebook using a token given by us. Once this is done, the use can make his own Sage worksheets and do the exercises given below.

2. Using sage

The exercises below are meant to give a number theoretic introduction to the computer program Sage and the reader should read some tutorials about Sage (or Python) before he/she can do the exercises. There are some nice manuals for Sage online. A first tutorial is given in `http://www.sagemath.org/doc/tutorial/`, and it is recommended to partially do this tutorial if you have never seen Sage (or Python) before. Especially `http://www.sagemath.org/doc/tutorial/tour_numtheory.html` can be useful.

3. Homework

The homework for next week (November 30, 2010) is to mail the solutions to the problems to mkosters@math.leidenuniv.nl. The solutions should be written in LaTeX and the (cleaned up) worksheet(s) should be included in this email. Finally share the worksheet with the user mkosters.

The problems will increase in difficulty and all exercises are worth the same. The star problems might be a bit harder and are adviced to be done when the other questions are solved.

4. Problems

Problem 4.1 (Just some algebraic number theory functions)**.**
Let $K = \mathbf{Q}(\alpha)$ where α is a root of $x^5 + 2 \cdot x^4 + 5 \cdot x^3 + 3 \cdot x^2 + 4$.

i. First calculate some invariants of K: the ring of integers, the discriminant, the class group (including the generators and relations).
ii. Find the units in the ring of integers (don't forget the torsion part) and express $-43519/2 \cdot \alpha^4 + 23722 \cdot \alpha^3 + 52935/2 \cdot \alpha^2 - 59495/2 \cdot \alpha + 32122$ in terms of fundamental units and the torsion part.
iii. * Finally for $A = \mathbf{Z}[\alpha]$ find $A^\dagger$ and calculate the size and structure of $A^\dagger/A$. What can you say about $[\mathcal{O}_K : A]$?

Problem 4.2 (Primes)**.**

i. Using elementary number theory it is relatively easy to show that Fermat's theorem holds for regular primes. These are primes p such that p does not divide the class number of $\mathbf{Q}(\zeta_p)$. Compute the first few regular primes.

ii. * A classical problem in number theory is to ask which primes can be written as $p = x^2 + ny^2$ where n is fixed and $x, y \in \mathbf{Z}$. Investigate this problem when $n = 27$ and conclude with a hypothesis (an if and only if statement). (The problem is 'completely' solved using Class Field Theory).

Problem 4.3 (Class group)**.**

i. Calculate the class number for $\mathbf{Q}(\sqrt{-d})$ where $2 \leq d \leq 300$ and d squarefree. Find all d such that this class number is 1.
ii. The same question as above where $-300 \leq d \leq -2$. Do you notice any difference about these class numbers (hint: make a plot, *list_plot*)?

Problem 4.4 (Exercise 5.34 - revised)**.** Let $K_1 = \mathbf{Q}(\sqrt{d_1})$ and $K_2 = \mathbf{Q}(\sqrt{d_2})$ be distinct real quadratic fields. Define $K_3 = \mathbf{Q}(\sqrt{d_1 d_2})$ and $K = K_1 K_2$.

i. During one of the exercise sessions, it was shown that $[\mathcal{O}_K^* : \mathcal{O}_{K_1}^* \mathcal{O}_{K_2}^* \mathcal{O}_{K_3}^*] \leq 8$. But what values for this index occur in nature? Try a reasonable number of cases and report your conclusions.
ii. * We can of course also look at the problem in an additive way. What values for $[\mathcal{O}_K : \mathcal{O}_{K_1} + \mathcal{O}_{K_2} + \mathcal{O}_{K_3}]$ occur? What is your hypothesis?