Sage homework - Mastermath Algebraic Number Theory 2014

November 26, 2014

Warning

You are responsible for making backups. Do this by downloading an `.sws` file of your worksheet by clicking "File..." and "Download worksheet to a file..."

How to hand in this homework:

You can hand in this homework in pairs. If you do, please write both of your names and student numbers clearly on everything you hand in.

In order for the assistants to check your results, you must hand in your Sage worksheet, and a written version of your solutions. For this, you should do *all three* of the following:

1. Email an `.sws` file of your Sage worksheet *with all output removed* to `ant.mm.2014@gmail.com`,
2. Share your Sage worksheet on the server with the user `antmm2014`.
3. Either make your worksheets as above into complete, readable solutions to the problems by inserting text, or hand in solutions to the problems on paper or pdf as usual. If you choose the former, then tell the assistants about this in your email and in your worksheet.

Regarding your worksheet: make sure it only contains the commands that you actually use in your solution, in the correct order, and provide cross-references where applicable.

Problem 1 (This problem consists of two unrelated questions)**.**

1. Find non-zero integers k and l satisfying $(161201\sqrt{101} + 1620050)^k = (64802401\sqrt{101} - 651256070)^l$.
2. Find the smallest pair of positive integers (x, y) such that $x^2 - 1621y^2 = 1$.

The other problems are on the next page!

Problem 2.

1. Compute *without proof* the structure and generators of the class group of $\mathbf{Q}(\zeta_n)$ for $n = 2, 4, 6, 8, \ldots, 60$.

2. What do you observe about the relation between the structure of the class group and how hard it is to compute it?

3. How many can you prove using Sage? [don't spend too much computation time on the last question: others are using this machine too]

Problem 3 (Based on exercises 16 and 17 of Chapter 7 of the notes)**.**

1. Let $L = \mathbf{Q}(\alpha)$ where $\alpha^3 + \alpha + 1 = 0$. Find the 30 smallest prime numbers that split completely in L.

2. Find the 30 smallest prime numbers of the form $x^2 + 31y^2$.

3. Formulate a conjecture based on 1 and 2. [This is a special case of *class field theory*]

Problem 4. Let $K = \mathbf{Q}(\beta)$ where β is a root of $(x^5 - 5x^2 - 25x + 1)$.

1. Compute a system of fundamental units of K.

2. Compute the class group of K.

3. For each ideal I of norm ≤ 11, express the class $[I]$ in terms of the generator(s) found in part 2.

4. Express the following element of K in terms of the units computed in part 1:

$$(-4574929890500\beta^4 + 5400980402504\beta^3 - 7568189710845\beta^2$$

$$+55244137453268\beta - 2180771968584)$$

Let $L = \mathbf{Q}(\alpha)$, where α is a root of $(x^5 - 10x^2 - 4x + 160)$.

5. Compute a system of fundamental units of the non-maximal order $\mathbf{Z}[\alpha]$.