EXERCISE 1.9

$$S = \varprojlim S_i$$

(A) Suppose that all the sets S_i are endowed with a compact Hausdorff topology, that all S_i are non-empty, and that all maps f_{ij} are continous. Prove that S is non-empty and compact.

$$j \geq i \quad S_{ij} := \{ (x_i) \in \prod S_i \mid f_{ij}(x_j) = x_i \}$$

↳ this is closed since def. by this cont. function

$S = \bigcap_{j \geq i} S_{ij}$ ⇝ closed since inters. of closed sets

$\Rightarrow \underset{\text{CLOSED}}{S} \subseteq \prod S_i$ ↘ COMPACT THANKS TO TYKHONOFF $\Rightarrow$ closed in a compact is compact!

Suppose $S = \bigcap_{j \geq i} S_{ij} = \emptyset$ $\Rightarrow$ by FINITE INTERSECTION PROP. of a compact it means that

$\exists$ fin. set $\{(j_1, i_1), \dots, (j_n, i_n)\}$ /

$$S_{i_1 j_1} \cap \dots \cap S_{i_n j_n} = \emptyset$$

Thanks to the "directed" property we can find $M \in I$ / $j_k \leq M$ $i_k \leq M$ $\forall k$

Now take $a_M \in S_M$ and def $a_{j_k} = f_{M j_k}(a_M)$

$a_{i_k} = f_{M i_k}(a_M)$

$\forall$ other $j \in I$ choose $a_j \in S_j$ arbit. ⇝ consider $a := (a_j)$ def. in this way

$\Rightarrow$ Now it's easy to prove that $a \in S_{i_k j_k}$ $\forall k = 1, \dots, n$.

$$\Rightarrow a \in S_{i_1 j_1} \cap \dots \cap S_{j_n i_n} = \emptyset \quad ↯ \quad \boxtimes$$

(B) $\forall S_i \neq \emptyset$ and finite $\Rightarrow S \neq \emptyset$

Use (A) given S_i the discrete topology.

(Raymond V.B.)

(C) Suppose that I is countable, that all S_i are non-empty and that all maps f_{ij} are surjective. Prove that $S \neq \emptyset$.

$\psi : \mathbb{N} \longrightarrow I$ $\leadsto$ we define three seq. $(M_i)_{i\in\mathbb{N}} \in \mathbb{N}$

$(m_i)_{i\in\mathbb{N}} \leadsto m_i \in S_{\psi(M_i)}$

$(a_i)_{i\in\mathbb{N}} \leadsto a_i \in S_{\psi(i)}$

- $M_1 = 1$
  $m_1 = a_1 \in S_{\psi(1)}$ arbitrarily.
- If we know m_K and M_K $\leadsto$ $M_{K+1} \in \mathbb{N}$ / $\psi(M_{K+1}) \geq \psi(M_K)$
  $\psi(M_{K+1}) \geq \psi(K+1)$

  $m_{K+1} \in S_{\psi(M_{K+1})}$ / $f_{\psi(M_{K+1})\psi(M_K)}(m_{K+1}) = m_K$
  (use of surj.)

  $\leadsto$ $a_{K+1} := f_{\psi(M_{K+1})\psi(K+1)}(m_{K+1})$
- One proves by induction that $i,j \in \mathbb{N}$ with $j \geq i$

  $\Rightarrow$ $\psi(M_j) \geq \psi(i)$ and $f_{\psi(M_j)\psi(i)}(m_j) = a_i$
- Finally $(a_i)_{i\in\mathbb{N}} \in S$

  i,j with $\psi(i) \geq \psi(j)$ $\quad N = \max\{i,j\}$

$$f_{\psi(i)\psi(j)}(a_i) = f_{\psi(i)\psi(j)}\left(f_{\psi(M_N)\psi(i)}(m_N)\right) \overset{!}{=} f_{\psi(M_N)\psi(j)}(m_N) = a_j$$

$\Rightarrow S \neq \emptyset$ $\boxtimes$

(D) All easy except to prove that $S = \emptyset$

$p : \mathbb{R} \longrightarrow \mathbb{Z}$ $\quad$ this is inj.

$r \longrightarrow p_{\{r\}}(r)$

$\Rightarrow$ inj of $\mathbb{R}$ into $\mathbb{Z}$ $\lightning$

$\boxtimes$

Exercise 10

If π_j is a profinite group for each j in a set $J \Rightarrow \prod_{j\in J} \pi_j$ is a prof. grp

It's suff. to use the characterisation of prof. grps which is given by denstra

PROF. GRP $\Longleftrightarrow$ TOP. GROUP COMPACT & TOT. DISCONNECTED

- A product of top. group is a top. grp $\Rightarrow \prod_{j\in J} \pi_j$ top. grp
- Tychonoff says that $\prod_{j\in J} \pi_j$ is COMPACT
- A product of tot. disc. is tot. disc. $\Rightarrow \prod_{j\in J} \pi_j$ tot. disc.

$\Rightarrow \prod_{j\in J} \pi_j$ PROF. GRP

Exercise 11

Let $\pi = \varprojlim \pi_i \subset \prod_{i\in I} \pi_i$ be a prof. grp with all π_i finite groups and $f_j : \pi_j \longrightarrow \pi$ the projection maps. Let further $\pi' \subset \pi$ be a subgroup.

(A) π' OPEN $\overset{①}{\Longleftrightarrow}$ π' CLOSED OF FIN. INDEX $\overset{②}{\Longleftrightarrow}$ $\exists j \in J : \operatorname{Ker} f_j \subset \pi'$

($\overset{①}{\Rightarrow}$) $\pi \setminus \pi' = \bigcup_{x\in \pi\setminus\pi'} x\pi'$ $\rightsquigarrow$ union of open is open and this union has to be finite since π is compact $\Rightarrow \pi'$ closed and of fin. index

($\overset{①}{\Leftarrow}$) π' closed $\Rightarrow$ also $x\pi'$ is closed using prop of top. grp

$\Rightarrow \pi\setminus\pi' =$ finite union of closed $\Rightarrow$ closed $\Rightarrow \pi'$ open

($\overset{②}{\Rightarrow}$) $\operatorname{Ker} f_j$ form a fund. system of neighborhoods of 1

$\Rightarrow$ ~~s~~ince $\pi' \ni 1$ and it's open we have $\exists j$ with $\operatorname{Ker} f_j \subset \pi'$

($\overset{②}{\Leftarrow}$) a neighborhood of 1 is contained in π' $\Rightarrow$ by multiplication any point of π' has a neighborhood in π' (using subgrp property)

$\Rightarrow \pi'$ open

(B) TFAE

i) π' closed

ii) $\exists (p_i \subset \pi_i)_{i\in I}$ subgrps s.t. $\pi' = \pi \cap \left(\prod_{i\in I} p_i\right)$

iii) $\exists$ " " with $f_{ij}[p_i] = p_j \ \forall i \geq j$

i) $\Rightarrow$ iii) Use $p_i = f_i(\pi)$. $\Rightarrow \pi' \subset \pi \cap \left(\prod_{i\in I} p_i\right)$

(VALENTIN Z.)

It is easy to see that $f_{ij}(p_i) = p_j \ \forall i \geq j$

$\pi \cap \left(\prod p_i\right) \subset \pi'$ since if we act by contrad. so assume that $\exists\ x = (x_i) \in \bigcap f_i^{-1}(p_i)$ and $x \notin \pi'$

$\Rightarrow$ ~~since~~ $\pi\setminus\pi'$ is open

$x \in \underbrace{\bigcap_{j\in J'} f_j^{-1}(L_j)}_{\text{basis elem. for the top.}} \subset \pi\setminus\pi'$ with J' finite $\subset I$ and $L_j \subset \pi_j$

Instead of taking L_j we can take the singletons $\{x_j\}$ $\Rightarrow$ $\bigcap_{j\in J'} f_j^{-1}(x_j) \subset \Pi \setminus \Pi'$

Let $l \in I$ s.t. $l \geq j \; \forall j \in J'$

$$\Rightarrow f_l^{-1}(\{x_l\}) \subset \bigcap f_j^{-1}(\{x_j\}) \subset \Pi \setminus \Pi'$$

$\lightning$ since $x_l \in f_l(\Pi')$.

$(iii) \Rightarrow (ii)$ trivial

$(ii) \Rightarrow (i)$ $\quad \Pi \cap \left(\prod_{i\in I} p_i\right) = \bigcap_{i\in I} f_i^{-1}(p_i)$ inters. of closed $\Rightarrow$ closed

(C) Π' PROFINITE $\Leftarrow \Pi'$ closed

$\left.\begin{array}{l} \Pi' \text{ closed in a compact} \Rightarrow \text{compact} \\ \Pi' \subset \Pi \text{ top. group tot. disc.} \Rightarrow \Pi' \text{ top. grp tot. disc.} \end{array}\right\} \Rightarrow$ PROF. grp

(D) Π' closed normal subgrp $\Rightarrow$ Π/Π' profinite with the quot. topology

property of top group $\leadsto$ Π/Π' is a top. grp

$\Pi \longrightarrow \Pi/\Pi'$ is a cont. map thanks to the def. of the quot. top.

Π/Π' is the image of a continous map from a compact space $\Rightarrow$ compact

It can be proved also that Π/Π' is tot. discon.

With an equivalent def of TOT. DISCON. we have to show that for any $x \in \Pi$ we get $\{x\Pi'\} = \bigcap_\lambda U_\lambda$, where U_λ is the family of compact open neighborhoods of $x\Pi'$ in Π/Π'.

We prove that its complement is not in this intersection.

Given $y\Pi'$ not in $\{x\Pi'\}$ $\leadsto$ $1 \notin x^{-1}y\Pi'$ $\Rightarrow$ $\exists\, U$ open subgrp disj. from $x^{-1}y\Pi'$

$$s : \Pi \longrightarrow \Pi/\Pi' \text{ continous (open)}$$

$s(xU)$ compact neighb. of $x\Pi'$ (it was a neig. of 1) in Π/Π'

" continous image of a clopen

$y\Pi'$ not in $s(xU)$ (def. of U) one of the U_λ

$\Rightarrow y\Pi'$ not in $\bigcap U_\lambda$

$\boxtimes$

Exercise 12

Ⓐ Let G be a group and $\hat{G}$ its profinite completion.
Prove that there is a natural group homomorphism $f: G \to \hat{G}$ for which $f(G)$ is dense in $\hat{G}$

$$\begin{aligned} \Theta_N : \ & G \longrightarrow G/N \quad \text{proj.} \\ & g \quad\quad\ \ \ gN \end{aligned}$$

$$\begin{aligned} f : \ & G \longrightarrow \prod G/N \\ & g \quad\quad\ \ \prod \Theta_N(g) \end{aligned} \qquad f(g) \in \hat{G} \text{ by the def of the proj.}$$

$f(G)$ is dense since Θ_N is surj. $\forall N$ and looking at the top. in $\prod G/N$

Ⓑ If G is a free group $\Rightarrow$ $f: G \to \hat{G}$ from Ⓐ is injective

It can be shown that a free group is residually finite (look some classical ref. of group theory)

$$\bigcap_{\substack{N \trianglelefteq G \\ \text{of fin. index}}} N = 1 \implies f: G \to \hat{G} \text{ is INJECTIVE}$$

Ⓒ See SERRE - TREES PROP. 6 pag. 9

$\hat{G}$ is trivial since any normal subgrp of fin index is trivial

EXERCISE 13

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$$

in $\mathbb{Z}/p^n\mathbb{Z}$ we take repres. between 0 and p^n-1 (it can be done)

Ⓐ $\mathbb{Z}_p^\times = \mathbb{Z}_p - p\mathbb{Z}_p$

$(\subseteq)$ $ab=1$ $\quad a=(a_0,a_1,\dots)$ $\quad b=(b_0,b_1,\dots)$ $\quad a_0b_0\equiv 1\,[p] \Rightarrow a_0\not\equiv 0\,[p]$

$a_nb_n\equiv 1\,[p^{n+1}]$

$(\supseteq)$ $a\in\mathbb{Z}_p-p\mathbb{Z}_p$ $\quad a=(a_0,a_1,\dots)$ with $a_0\not\equiv 0\,[p]$

$\Downarrow$

$a_n\not\equiv 0\,[p]$

$\exists\, b_0 \,/\, a_0b_0\equiv 1\,[p]$

$\Swarrow$

$\forall n\ \exists\, b_n$ with $a_nb_n\equiv 1\,[p^{n+1}]$

and $a_nb_n\equiv a_{n-1}b_{n-1}\,[p^n]$

but $a_n\equiv a_{n-1}\,[p^n] \Longrightarrow b_n\equiv b_{n-1}\,[p^n]$

$\Rightarrow$ $b:=(b_0,b_1,\dots,)$ is $\in\mathbb{Z}_p$ and $ab=1$

Ⓑ $a\in\mathbb{Z}_p\setminus\{0\}$ can be uniquely written as $a=up^n$ with $u\in\mathbb{Z}_p^\times$, $n\in\mathbb{Z}$ $n\geq 0$.

If a is a unit we're done

Suppose $a_0\equiv 0\,[p]$ $\quad\exists$ smallest $n \,/\, a_n\not\equiv 0\,[p^{n+1}]$

$a_{n+s}\equiv a_{n-1}\equiv 0\ [p^n]$ $\quad s\geq 0$

$\Rightarrow$ we consider $b=\Big(\underbrace{\frac{a_n}{p^n}}_{y_0},\underbrace{\frac{a_{n+1}}{p^n}}_{y_1},\dots\Big)$

$y_s := \dfrac{a_{n+s}}{p^n} \quad \rightsquigarrow \quad y_s\not\equiv 0\,[p]$

$p^n y_s - p^n y_{s-1} = a_{n+s}-a_{n+s-1}\equiv 0\ [p^{s+n}]$

$\Rightarrow y_s\equiv y_{s-1}\,[p^s] \Rightarrow b\in\mathbb{Z}_p$ and $a=p^nb$

The repres. is unique $\to$ the exp of p has to be the same otherwise

$p^m u = p^n v$ with $m\geq n \Rightarrow p^{m-n}\dfrac{u}{v}$ is a unit ↯

$\Rightarrow m=n$

$\Rightarrow u=v$

$\to$ ~~Suppose $a=p^m u=p^m v \Rightarrow p^m u_s = p^m v_s\ [p^{s+1}]$~~

~~where $p\nmid u_s$ and v_s~~

~~$s=m \to p^m u_m = p^m v_m\ [p^{m+1}]$~~

~~$u_m\equiv v_m\ [p]$ $\quad u_s\equiv u_{m+s}\equiv v_{m+s}\equiv v_s\ [p^{s+1}]$~~

~~$\Rightarrow u=v$~~

Ⓒ $\mathbb{Z}_p$ is a local domain with res. class field $\mathbb{F}_p$

domain $\rightsquigarrow$ $p^m u\cdot p^k v=0 \longrightarrow p^{m+k}uv=0 \Rightarrow p^{m+k}=0$ ↯

$p\mathbb{Z}_p$! maximal ideal (since any elem. not in it is a unit)

$\mathbb{Z}_p/p\mathbb{Z}_p \xrightarrow{\sim} \mathbb{F}_p$ looking at a_0

EXERCISE 1.14

$$\boxed{\hat{\mathbb{Z}} \simeq \prod_{p \text{ prime}} \mathbb{Z}_p}$$

$x \in \hat{\mathbb{Z}}$ $\quad x = (x_n)$ $\quad$ since $\mathbb{Z}/n\mathbb{Z} \simeq \prod_{p_i | n} \mathbb{Z}/_{p_i^{\operatorname{ord}_{p_i}(n)}}\mathbb{Z}$

by the Chinese Reminder Theorem

any x_n is def. by $\left(x_{p_i^{\operatorname{ord}_{p_i}(n)}} \right)$ using the restriction map

we have

$$(x_n) \longrightarrow \prod_p \underbrace{(x_p, x_{p^2}, \dots)}_{\in \mathbb{Z}_p}$$

since in $\hat{\mathbb{Z}}$ $x_{p^i} \equiv x_{p^j} [p^j]$

* clearly injective & surjective
* to prove continuity take $H \subset \prod_{p \in P} \mathbb{Z}_p$ open

$\Rightarrow \exists$ $Q \subset P$ finite s.t. $H = \left(\prod_{q \in Q} H_q \right) \times \left(\prod_{p \in P \setminus Q} \mathbb{Z}_p \right)$

with $H_q \subset \mathbb{Z}_q$ open $\forall q \in Q$

$\Rightarrow$ $\varphi^{-1}(H)$ is given by a fin. number of conditions $\Rightarrow \varphi^{-1}(H)$ open in $\hat{\mathbb{Z}}$

$\Rightarrow \varphi$ continous

* Since $\hat{\mathbb{Z}}$ is compact and $\prod \mathbb{Z}_p$ Hausdorff we have that also the inverse is continous $\Rightarrow$ $\simeq$ of top. space
* Easy to verify using morphism property.

EXERCISE 1.15

$$\mathbb{Z}_{10} = \varprojlim \mathbb{Z}/10^n\mathbb{Z}$$

Ⓐ Prove that each $a \in \mathbb{Z}_{10}$ has a unique repres. $a = \sum_{n=0}^{\infty} c_n 10^n$ with $c_n \in \{0,1,..,9\}$.

$x = (x_0, x_1, \dots) \in \mathbb{Z}_{10}$ we can consider $\bar{x}_i \equiv x_i \ [10^{i+1}]$ with $0 \le \bar{x}_i < 10^{i+1}$

$x = \bar{x} = (\bar{x}_0, \bar{x}_1, \dots)$ $\qquad c_0 := \bar{x}_0$

$\underbrace{\bar{x}_i}_{< 10^{i+1}} = \underbrace{\bar{x}_{i-1}}_{< 10^i} + c_i 10^i \implies c_i \in \{0,1,..,9\}$

$\implies$ by induction $\bar{x}_i = c_0 + c_1 \cdot 10 + c_2 \cdot 10^2 + \dots + c_i \cdot 10^i$

$\implies$ to any x we associate $c_0 + c_1 10 + \dots$

This repr. is ! since if $x = \underbrace{\underbrace{d_0}_{\bar{y}_0} + d_1 \cdot 10}_{\bar{y}_1} + d_2 \cdot 10^2 + \dots$

$\implies \exists$ smallest i / $c_i \ne d_i$

$\implies \bar{x}_i = c_0 + c_1 \cdot 10 + \dots + c_i 10^i \equiv d_0 + d_1 10 + \dots + d_i \cdot 10^i \ [10^{i+1}]$

$\implies (c_i - d_i)\, 10^i \equiv 0 \ [10^{i+1}] \implies c_i \equiv d_i \ [10]$ ↯ ☒

Ⓑ Prove that there exists a ! continous function $v : \mathbb{Z}_{10} \to \mathbb{R}$ s.t. $v(a) = (\#\text{ of factors } 2 \text{ in } a)^{-1}$ for each positive integer a.

Problem in the def since $v(\text{odd}) = 0^{-1}$?

$\implies$ new def for v (similar to the previous one) $v(a) = (1 + \#\text{of factors } 2 \text{ in } a)^{-1}$

m be the function sending $(a_n)_{n \ge 1} \in \mathbb{Z}_{10} \longrightarrow$ least n / $a_n \not\equiv 0 \ [2^n]$

$v(a) := (1 + m(a))^{-1}$ where $\infty^{-1} = 0$.

Image $\longrightarrow S := 0 \cup \{\frac{1}{k} : k \in \mathbb{N}\}$ with a basis for the top.

$C := \{(-\varepsilon, \varepsilon) \cap S\} \cup \{\frac{1}{k}\}$

$v^{-1}(\{\frac{1}{k}\})$ is CLOPEN since def. by a fin. number of conditions

$v^{-1}((-\varepsilon, \varepsilon) \cap S) = 2^N \mathbb{Z}_{10}^*$ which is open for some $N / \frac{1}{N} < \varepsilon$ $\implies v$ continous

To prove that it's unique it's suf. to see that $\mathbb{N}$ is dense in $\mathbb{Z}_{10}$.

(15 C) Let $(a_n)_{n=0}^{\infty}$ be a sequence of pos. integers not div. by 10 / the number of factors 2 in a_n tends to infinity for $n \to \infty$. Prove that the sum of the digits of a_n in the decimal system tends to infinity.

If the sum of the digits doesn't conv. to ∞

$\Rightarrow$ it conv. to a fin. number (is a increasing sequence) $\Rightarrow$ constantly 0 after a N

$\Rightarrow$ a_n has to conv. to an element in $\mathbb{N}$ called b

$$\Rightarrow \underset{\substack{| \\ \text{continous } f.}}{\lim v(a_n)} = \lim v(b) = \text{fin. numb.} \Rightarrow ⚡ \qquad \boxtimes$$

ANOTHER SOLOTION FOR B)

$$\begin{tikzcd}
\mathbb{Z}_{10} \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \arrow[rr] \arrow[d, "\pi_2 \text{ proj.}"] & & \frac{1}{k+1} \text{ in } \mathbb{R} \\
\mathbb{Z}_2 \arrow[r, "\text{valuation}"'] & \{0,1,\dots\} \cup \{\infty\} \arrow[ur, "k \mapsto \frac{1}{k+1}"'] &
\end{tikzcd}$$

all is continous

EXERCISE 16

Ⓐ Prove that each $a \in \hat{\mathbb{Z}}$ has a ! rep. $a = \sum_{n=1}^{\infty} c_n n!$ with $c_n \in \{0, \dots, n\}$

ex. 1.17 c and do as in ex. 1.15 a

Ⓑ Let $b \in \mathbb{Z}$, $b \geq 0$ and define the seq. $(a_n)_{n=0}^{\infty}$ of non-neg. integers by $a_0 = b$, $a_{n+1} = 2^{a_n}$. Prove that $(a_n)_{n=0}^{\infty}$ converges in $\hat{\mathbb{Z}}$ and that $\lim_{n \to \infty} a_n \in \hat{\mathbb{Z}}$ is ind. of b.

$$(a_n)_{n=0}^{\infty} \text{ conv. in } \hat{\mathbb{Z}} \iff (a_n)_{n=0}^{\infty} \text{ conv. mod } n\mathbb{Z} \ \forall n$$

mod 2 the seq. is constantly 0 → no dipendence from b here

mod 3 $b, 2^b, 2^{2^b}, 2^{2^{2^b}}$ conv. since the seq. of exp. ~~conv. mod~~ is constant mod $\varphi(3) = 2$

⇓ for all the other n

... and so on by induction, looking at the exponent mod $\varphi(n)$

Jacopo 6.)

Ⓒ Let $a = \lim_{n \to \infty} a_n$ as in (b) and write $a = \sum_{n=1}^{\infty} c_n n!$ as in Ⓐ. Compute c_n for $1 \leq n \leq 10$.

We choose $b = 1$. Then

$$\begin{aligned}
a_0 &= 1 \longleftrightarrow (1, 0, \dots) \\
a_1 &= 2 \longleftrightarrow (0, 1, 0, \dots) \\
a_2 &= 4 \longleftrightarrow (0, 2, 0, \dots) \\
a_3 &= 16 \longleftrightarrow (0, 2, 2, 0, \dots) \\
a_4 &= 65536 \longleftrightarrow (0, 2, 2, 0, 0, 0, 5, 1, 0, \dots) \\
a_5 &= 2^{65536} \equiv 26676736 \ [11!] \longleftrightarrow (0, 2, 2, 0, 0, 0, 5, 4, 3, 7, \dots) \\
a_6 &= 2^{a_5} \equiv 26676736 \ [11!] \longleftrightarrow (\text{''} \qquad \text{''})
\end{aligned}$$

and so on $\Rightarrow a_n$ has the same last 10 digits as a_5 $\forall n \geq 5$ → the last 10 digits are

Exercise 17

(A) If J is a cofinal subset of a directed part. ord. set $\Rightarrow$ J is directed.

Easy

(B) $$\boxed{\varprojlim_{j\in J} S_j \simeq \varprojlim_{i\in I} S_i}$$

$$(x_j)_{j\in J} \longrightarrow (x')_{i\in I} \text{ where } \left.\begin{array}{l} \text{if } i\in J \quad x_i' := x_i \\ \text{if } i\notin J \text{ take } k\geq i \text{ with } k\in J \\ \qquad f_{k,i}(x_k) =: x_i' \end{array}\right\} \in \varprojlim_{i\in I} S_i \text{ easy}$$

- injective since in the place $\in J$ we have 1
- surjection we take $(x_j) \in \prod S_j$ looking at the place in J

$\boxtimes$

(C) $$\boxed{\hat{\mathbb{Z}} \simeq \varprojlim \mathbb{Z}/n!\,\mathbb{Z}}$$

$J = \{n!\}_{n\in\mathbb{N}}$ is cofinal for $I = \{n \in \mathbb{N}\}$.