HOMEWORK SET 3

Local Class Field Theory - Fall 2011

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Exercise 3.1. Suppose A is an abelian group which is torsion (every element has finite order). Show that $\mathrm{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z}) \cong \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})$. Hint: use an injective resolution of $\mathbb{Z}$.

Proof. We take the usual injective resolution (of abelian groups) of $\mathbb{Z}$ given by

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \longrightarrow \dots \quad .$$

The sequence is clearly exact and all the abelian groups involved (except $\mathbb{Z}$ of course) are injective, since they are divisible and we are working in the category of $\mathbb{Z}$-modules. (Here $\mathbb{Q}$ is considered as a group under the additive law).
Thus to compute $\mathrm{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z})$, we have to apply the left exact covariant functor $Hom(A,-)$, getting

$$Hom(A,\,\mathbb{Q}) \xrightarrow{d^0} Hom\,(A,\,\mathbb{Q}/\mathbb{Z}) \xrightarrow{d^1} 0 \longrightarrow \dots \quad .$$

Since A is a torsion group we see that $Hom(A,\,\mathbb{Q}) = 0$, since $\mathbb{Q}$ has no torsion points (except 0).
Finally, just by definition, we see that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z}) = \frac{ker(d^1)}{Im(d^0)} \cong \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z}).$$

□

Exercise 3.2. Given a prime p. Explicitly list representatives of all isomorphism classes of extensions of the group $\mathbb{Z}/p\mathbb{Z}$ by the group $\mathbb{Z}/p\mathbb{Z}$. Here an exension is an exact sequence of groups of the form $0 \to \mathbb{Z}/p\mathbb{Z} \to X \to \mathbb{Z}/pZ \to 0$, and two of these are said to be isomorphic if there is a morphism of short exact sequences between them, where the map on the outer two $\mathbb{Z}/p\mathbb{Z}$ is the identity map. Hint: show that there are p of them.

Proof. ♡ **There are p of them.** It's well known that the group of isomorphism classes of extensions of the group $\mathbb{Z}/p\mathbb{Z}$ by the group $\mathbb{Z}/p\mathbb{Z}$ is isomorphic to $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$ (look at Wikipedia or at Weibel or at Hilton, Stammbach - A course in homological algebra). So to compute its cardinality we compute $|\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})|$.

In order to compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$, we take the following injective resolution of $\mathbb{Z}/p\mathbb{Z}$ (to prove injectivity and exactness look at ex. 1):

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Q}/p\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \longrightarrow \dots \quad .$$

We now apply the covariant functor $Hom(\mathbb{Z}/p\mathbb{Z}, -)$ (and we erase the first term), getting

$$Hom(\mathbb{Z}/p\mathbb{Z},\, \mathbb{Q}/p\mathbb{Z}) \xrightarrow{\ a\ } Hom\,(\mathbb{Z}/p\mathbb{Z},\, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\ b\ } 0 \longrightarrow \dots \quad ;$$

thus by definition we have

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \frac{ker(b)}{Im(a)} \cong \frac{Hom(\mathbb{Z}/p\mathbb{Z},\, \mathbb{Q}/\mathbb{Z})}{Im(a)}.$$

Now we see that

$$Hom(\mathbb{Z}/p\mathbb{Z},\, \mathbb{Q}/p\mathbb{Z}) \subseteq Hom(\mathbb{Z}/p\mathbb{Z},\, \mathbb{Z}/p\mathbb{Z}),$$

which thus becomes 0 if we apply a on it.
Finally, we see, in this way, that

$$\begin{aligned} |\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})| = |Hom(\mathbb{Z}/p\mathbb{Z},\, \mathbb{Q}/\mathbb{Z})| &= \left| Hom\left(\mathbb{Z}/p\mathbb{Z},\, \frac{1}{p}\mathbb{Z}/\mathbb{Z}\right)\right| \\ &= |Hom(\mathbb{Z}/p\mathbb{Z},\, \mathbb{Z}/p\mathbb{Z})| \\ &= p. \end{aligned}$$

♡ **Representatives of all isomorphism classes of extensions of the group $\mathbb{Z}/p\mathbb{Z}$ by the group $\mathbb{Z}/p\mathbb{Z}$.** We claim that the p extensions are of the following forms:

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0 \quad \text{(Split one)}; \tag{1}$$

$$\begin{array}{l} 0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{\ \phi_i\ } \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\ \pi\ } \mathbb{Z}/p\mathbb{Z} \longrightarrow 0\ , \text{ where } i \in \{1, \dots, p-1\}. \\ \qquad x \longmapsto ipx \\ \qquad\qquad y \longmapsto y \bmod p \end{array} \tag{2}$$

They are clearly p extension and we have just to check that they are not isomorphic each other.
First of all note that the extension (1) cannot be isomorphic to any of the second type, since $(\mathbb{Z}/p\mathbb{Z})^2 \ncong \mathbb{Z}/p^2\mathbb{Z}$.
The only remaining deal is to show that if we take two extensions of the second type, then they cannot be isomorphic. We proceed by contradiction, take two natural numbers i and j in $\{1, \dots, p-1\}$, such that $i \neq j$, and suppose that we have the following isomorphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{\ \phi_i\ } & \mathbb{Z}/p^2\mathbb{Z} & \xrightarrow{\ \pi\ } & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \\ & & \| & & f \downarrow \cong & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{\ \phi_j\ } & \mathbb{Z}/p^2\mathbb{Z} & \xrightarrow{\ \pi\ } & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0\ . \end{array}$$

Then since $\pi \circ f = \pi$, we have $f(1) \equiv 1 + pn \bmod p^2$ for some $n \in \mathbb{Z}$, which implies that $f(ip) \equiv ip \bmod p^2$. So if $f \circ \phi_i = \phi_j$, computing it in 1 we have $ip \equiv jp \bmod p^2$ which gives us the contradiction $i \equiv j \bmod p$.

♡ What does it happen if p is not prime?

□

Exercise 3.3. On the category of finite abelian groups, consider the bifunctors $\mathrm{Hom}(A, B)$ and $D(A \otimes D(B))$, where $D(A) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ denotes the dual of A. Show that both functors are left-exact in both A and B, contravariant in A and covariant in B. Show also that the two bifunctors are equivalent.

Proof. First solution (solution given by Maxim L. Mornev).

♡ **Bifunctors contravariant in A and covariant in B.** It's very easy and left to the reader.

♡ **Natural equivalence between the two bifunctors**

⋆ **Recall.** Remember that to show that two functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ (the same for bifunctors) are natural equivalent we need to prove two facts: given $C \in \mathcal{C}$ then $\mathcal{F}(C) \cong \mathcal{G}(C)$ and also that this isomorphism is natural, which means given $C \to C'$ in $\mathcal{C}$ we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(C) & \longrightarrow & \mathcal{G}(C) \\ \downarrow & & \downarrow \\ \mathcal{F}(C') & \longrightarrow & \mathcal{G}(C') \end{array} .$$

⋆ **Remark.** It's not difficult to see that when A is finite, then $D(A)$ is finite too, indeed if we denote by $(\mathbb{Q}/\mathbb{Z})[n]$ the set of elements in $\mathbb{Q}/\mathbb{Z}$ of order dividing n, we have $D(A) = Hom\,(A, (\mathbb{Q}/\mathbb{Z})[n])$, finite since $(\mathbb{Q}/\mathbb{Z})[n]$ is finite.

⋆ **Claim: the functor $D\,(D(-))$ is naturally equivalent to the idenity functor.** We consider the map $A \to D(D(A))$ which sends $a \mapsto (\phi \mapsto \phi(a))$. This is clearly a homomorphism and we show that it's injective.
Let a be an element in the kernel and consider the cyclic subgroup of A given by $\langle a \rangle$. Any homomorphism $\phi : \langle a \rangle \to \mathbb{Q}/\mathbb{Z}$ is 0 since it can always be extended to a homomorphism $\phi : A \to \mathbb{Q}/\mathbb{Z}$ (using the fact that $\mathbb{Q}/\mathbb{Z}$ is injective as an abelian group, since divisible). Thus the map $\phi : \langle a \rangle \to \mathbb{Q}/\mathbb{Z}$ defined by sending $a \mapsto \frac{1}{ord(a)}$ has to be 0, which implies that $\frac{1}{ord(a)} \in \mathbb{Z} \Rightarrow ord(a) = 1 \Rightarrow a = 1$, which proves that our original map is really an injection.

To show that our map is an isomorphism we look at the cardinality of our sets and in particular it's not difficult to see that $\sharp D(A) \leq \sharp A$ (use the fundamental Theorem of finite abelian groups and show it just for cyclic groups), which implies that (applying again the dual) $\sharp D(D(A)) \leq \sharp A$. So we see in particular that our injection has to be really an isomorphism.

The isomorphism $A \cong D(D(A))$ is natural. Given a homomorphism of abelian groups $\psi : A \to B$ then the dual homorphism $D(\psi) : D(B) \to D(A)$ is given by $\phi \mapsto \phi \circ \psi$; so if we apply the dual again we have that $D(D(\psi)) : D(D(A)) \to D(D(B))$ works as $\phi \mapsto \phi \circ D(\psi)$. It's immediate to see that the following diagram commutes

$$\begin{array}{ccc} A & \longrightarrow & D(D(A)) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle D(D(\psi))} \\ B & \longrightarrow & D(D(B)) \end{array} .$$

⋆ **Remark.** Also when A is not finite, we have always an injection of $A \hookrightarrow D(D(A))$.

⋆ **Conclusion of the proof.** We recall that, just by the definition of the tensor product, we have that $Hom(A \otimes B,\, C)$ is naturally isomorphic to $Hom(A,\, Hom(B,\, C))$. Thus we have

$$\begin{aligned} D(A \otimes D(B)) &= Hom(A \otimes Hom(B,\, \mathbb{Q}/\mathbb{Z}),\, \mathbb{Q}/\mathbb{Z}) \\ &\cong Hom(A,\, Hom(Hom(B,\, \mathbb{Q}/\mathbb{Z}),\, \mathbb{Q}/\mathbb{Z})) \\ &= Hom(A,\, D(D(B))) \\ &\cong Hom(A,\, B)\ ; \end{aligned}$$

where the symbol $\cong$ means natural isomorphic.

♡ **Left exactness.** As we have recalled in class, the functor $Hom(A,\, B)$ is left exact in both variables (Ref. Atiyah-McDonald "Introduction to Commutative Algebra", Prop. 2.9), so, thanks to our natural isomorphism, also $D(A \otimes D(B))$ has the same property.

Second Solution (solution given by Raymond van Bommel and Milan Lopuhaä).
There was also a more direct way to show the natural isomorphism, supplying explicitly the isomorphism between the two bifunctors. The map is easily defined in the following way

$$\begin{aligned} Hom(A,\, B) &\longrightarrow D(A \otimes D(B)) \\ f &\longmapsto (a \otimes b' \mapsto b'(f(a)))\ . \end{aligned}$$

Using the fundamental Theorem of finite abelian groups, it's not too difficult to construct the inverse of our map, what is it? □

Exercise 3.4. Of the 64 bifunctors mentioned in class, which other ones are also left-exact in both A and B, contravariant in A and covariant in B? Can you show that they are all equivalent? Recall, we get to 64 by considering Hom, $\otimes$, Ext and Tor (factor 4) taken duals of A, B and the result (factor 8) and switching A and B (factor 2).

This exercise is not so easy and in particular not so short, there's some work behind (this explains also why nobody has tried to solve it). So let's say that we let it as an exercise for the spare time.

Exercise 3.5. For an abelian group A let $T(A)$ be the torsion subgroup of A, i.e., the subgroup consisting of all elements of finite order. Show that T is a left exact functor, and compute its derived functors. Hint: $T(A)$ is the kernel of $A \to A \otimes \mathbb{Q}$.

Proof. ⋆ **A reference for the proof of the fact that $T(A)$ is the kernel of $A \to A \otimes \mathbb{Q}$:** Keith Conrad's expository article "Tensor products", Corollary 4.22.

⋆ **T is a left exact functor.** It's easy to see that T is a functor (the image of a torsion element under a homomorphism has to be torsion) and also that it's covariant. Also the left exactness is not hard, indeed given the exact sequence

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0 ,$$

we have to show that

$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C)$$

is exact.

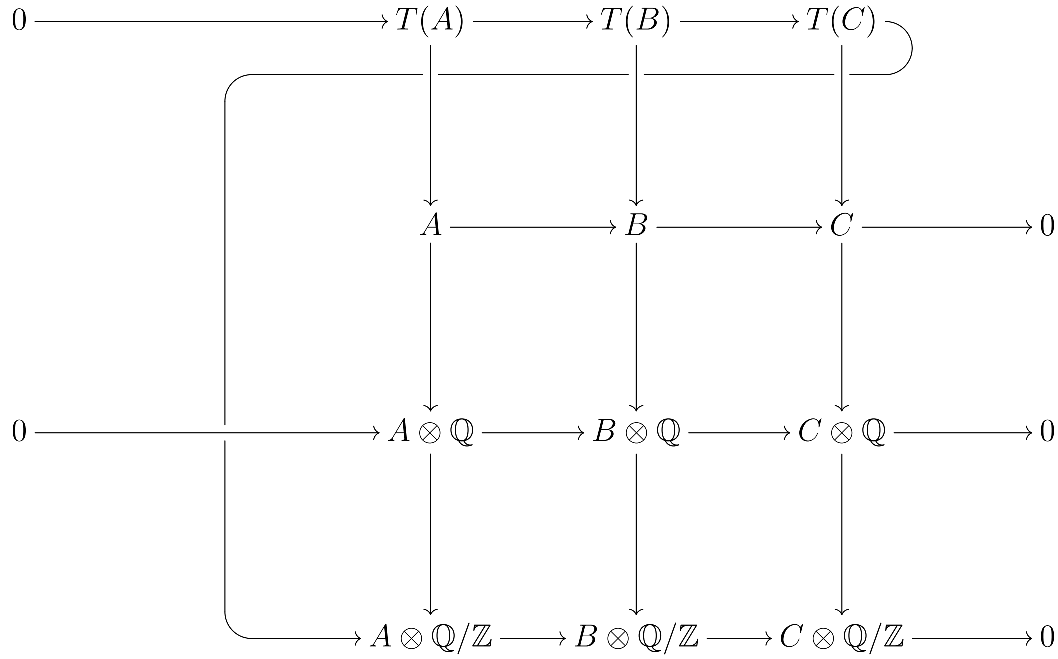
The fact that the first map is injective is clear since it's a restriction of ϕ which is injective. It's also clear that this is a complex, it remains just to show the exactness in $T(B)$. If $b \in T(B)$ is sent to 0 then $b = \psi(a)$ for some $a \in A$, we want to prove that actually $a \in T(A)$. Since ϕ is injective, the order of a is equal to the order of b and so we have that a does belong to $T(A)$.

⋆ **First solution of the computation of the derived functors (using the notion of universal δ-functor).**

We know that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, since torsion free (Mathworld link), so the covariant functor $- \otimes \mathbb{Q}$ is exact, so we can consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A \otimes \mathbb{Q} & \longrightarrow & B \otimes \mathbb{Q} & \longrightarrow & C \otimes \mathbb{Q} & \longrightarrow & 0 \end{array} .$$

Using the fact that $T(A)$ is the kernel of $A \to A \otimes \mathbb{Q}$ and that the cokernel of that map is $A \otimes \mathbb{Q}/\mathbb{Z}$ (look at A as $A \otimes \mathbb{Z}$), we can apply the Snake lemma getting



Thus we can define the collection of functors $F = (F^i)_{i \geq 0}$ on the category of abelian groups, where for any A abelian group

$$F^i(A) = \begin{cases} T(A) & \text{if } n = 0\,, \\ A \otimes \mathbb{Q}/\mathbb{Z} & \text{if } n = 1\,, \\ 0 & \text{for } n \geq 2\,. \end{cases}$$

We claim that this collection is really the right derived functors we were looking for and to prove it, we use the Remark $A.12$ of Milne (for a more general theorem using the notion of *effaceable functor* I suggest you to give a look at the first paragraph of Chap. 3 of Hartshorne's Algebraic Geometry - Theorem 1.3A). Looking at the remark we see that we have the proof of our claim if:

- ♣ $F^0(A) = T(A) \longrightarrow$ It's satisfied by the definition of F,
- ♣ $F^i(I) = 0$ for all $i \geq 1$ when I is injective $\longrightarrow$ In the category of abelian groups to be injective means to be divisible and, with an analogous proof as the one you will find in the next point, it's easy to see that if I is divisible then $I \otimes \mathbb{Q}/\mathbb{Z} = 0$.
- ♣ F is a δ-functor (look at the definition on Harthshorne or Weibel) $\longrightarrow$ It comes directly from its construction using the Snake Lemma.

⋆ **The best and clever solution of the computation of the derived functors (inspired by the solution given by Maxim L. Mornev).**

The basic idea of this solution is the following claim: given D a divisible group then the natural map $D \xrightarrow{\phi} D \otimes \mathbb{Q}$ is surjective. Let's prove it, given an element $d \otimes \frac{n}{m}$ by the bilinearity property of the tensor product we have $d \otimes \frac{n}{m} = nd \otimes \frac{1}{m}$ which thanks to the divisibility of D ca be written as $md' \otimes \frac{1}{m}$ which again by bilinearity is equal to $d' \otimes 1 \in \phi(D)$ as we wanted to prove.

The category of abelian group has enough injective (look at Weibel), so given an abelian group A, we can find an injection of $A \longrightarrow I$, where I is injective (= divisible, since we are in the category of abelian groups). Moreover if we take a subgroup $J \subset I$ we have that the quotient I/J is again divisible (easy proof), so in particular for any abelian group A we can always find an injective resolution

$$0 \longrightarrow A \xrightarrow{\psi} I \longrightarrow I' \longrightarrow 0 \ ,$$

where $I' = I/\psi(A)$. In particular from this we see that $R^nT(A) = 0$ for any $n \geq 2$ (this is true for any functor on the category of abelian groups).

It remains now to compute $R^1T(A)$ and to do it, we apply the functor $T(-)$ to the previous exact sequence getting the following complex (not exact)

$$T(I) \xrightarrow{t_0} T(I') \longrightarrow 0 \longrightarrow \dots \ ;$$

by definition we have

$$R^1T(A) = \frac{T(I')}{t_0\left(T(I)\right)}.$$

Now we use the long exact sequence obtained using the snake lemma in the previous solution, where now we take $B = I$ and $C = I'$. Using the initial claim of the solution we have $I \otimes \mathbb{Q}/\mathbb{Z} = I' \otimes \mathbb{Q}/\mathbb{Z} = 0$ and the long exact sequence becomes

$$0 \longrightarrow T(A) \longrightarrow T(I) \xrightarrow{t_0} T(I') \longrightarrow A \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \longrightarrow \dots,$$

from what we understand that $R^1T(A) = \dfrac{T(I')}{t_0\left(T(I)\right)} = A \otimes \mathbb{Q}/\mathbb{Z}.$

Summarizing, we have

$$R^nT(A) = \begin{cases} T(A) & \text{if } n = 0\,, \\ A \otimes \mathbb{Q}/\mathbb{Z} & \text{if } n = 1\,, \\ 0 & \text{for } n \geq 2\,. \end{cases}$$

□

Exercise 3.6 Show that $H^1(G,\mathbb{Z}) = 0$ for any finite group G, with trivial action on $\mathbb{Z}$. If G is cyclic of order 2, and $\mathbb{Z}^-$ is $\mathbb{Z}$ with non-trivial G-action, compute $H^1(G,\mathbb{Z}^-)$.

Proof. ♡ As we have seen in class, when the action of G is trivial, we know that

$$\mathrm{H}^1(G,\mathbb{Z}) = Hom(G,\mathbb{Z})$$

and $Hom(G,\mathbb{Z}) = 0$, since G is finite and the only finite subgroup of $\mathbb{Z}$ is $\{0\}$.

♡ If $G = \{1,\, \sigma\}$, with $\sigma^2 = 1$, the non trivial action of G on $\mathbb{Z}$ is given by $\sigma.x = -x$ for any $x \in \mathbb{Z}$. We recall that, as we have seen in class, a consequence of the Bar resolution is that

$$\mathrm{H}^1(G,\mathbb{Z}^-) = \frac{\{\text{crossed homomorphisms}\}}{\{\text{principal crossed homomorphisms}\}},$$

where a crossed homomorphism is a map $\phi : G \longrightarrow \mathbb{Z}^-$ such that

$$\phi(g_1 g_2) = g_2.\phi(g_1) + \phi(g_2) \text{ for all } g_1,\, g_2 \in G$$

and a principal homomorphism is a map sending $\sigma \longmapsto \sigma.x - x$, where x is a fixed element in $\mathbb{Z}$. With the given action it's easy to see that any map ϕ, sending $1 \longmapsto 0$ and $\sigma \longmapsto x \in \mathbb{Z}$, is a crossed homomorphism. While a principal homomorphism is a map sending $1 \longmapsto 0$ and $\sigma \longmapsto -2x$ where $x \in \mathbb{Z}$.
We conclude so, that

$$\mathrm{H}^1(G,\mathbb{Z}^-) = \frac{\{\text{crossed homomorphisms}\}}{\{\text{principal crossed homomorphisms}\}} \cong \mathbb{Z}/2\mathbb{Z}.$$

□