HOMEWORK SET 4

Local Class Field Theory - Fall 2011

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Exercise 4.1. For an abelian group A with endomorphisms $f, g : A \to A$ satisfying $fg = gf = 0$ we define

$$q_{f,g}(A) = [\mathrm{Ker} f : \mathrm{Im} g]/[\mathrm{Ker} g : \mathrm{Im} f]$$

whenever both indices are finite.

1. For $n \in \mathbb{Z}$ compute $q_{0,n}(\mathbb{Z})$.

2. Show that $q_{f,g}(A) = 1$ whenever A is finite.

3. Show how the usual Herbrand quotient $h(M)$ of a module M over a finite cyclic group G is a special case of this.

Proof. **1.** With $n \in \mathbb{Z}$ we just denote the automorphism in $\mathbb{Z}$ which sends $x \mapsto nx$, while 0 denotes the zero map. Now it's immediate to see that $\mathrm{Ker}(0) = \mathbb{Z}$ and $\mathrm{Im}(0) = \{0\}$, while $\mathrm{Ker}(n) = \{0\}$ and $\mathrm{Im}(n) = n\mathbb{Z}$. Thus we get

$$q_{0,n}(\mathbb{Z}) = [\mathbb{Z} : n\mathbb{Z}]/[\{0\} : \{0\}] = n.$$

2. When A is finite, we have that the Kernels and the Images of automorphisms have to be finite. Moreover given $H \subset G$ both finite groups, we have that $[G : H] = |G|/|H|$. Using this remark and the isomorphism Theorem ($A/\mathrm{Ker} f \cong \mathrm{Im} f$, $A/\mathrm{Ker} g \cong \mathrm{Im} g$) we have $|\mathrm{Im} g| = |A|/|\mathrm{Ker} g|$ and $|\mathrm{Ker} f| = |A|/|\mathrm{Im} f|$. Thus we get

$$q_{f,g}(A) = \frac{|\mathrm{Ker} f|/|\mathrm{Im} g|}{|\mathrm{Ker} g|/|\mathrm{Im} f|} = \frac{|A|}{|A|} = 1.$$

3. If G is generated by σ, take $f : M \to M$ given by $m \mapsto (\sigma - 1)m$ and $g = \mathrm{Nm}_G$, the norm map. Then by definition we have $\hat{H}^0(G, M) = \dfrac{\mathrm{Ker} f}{\mathrm{Im} g}$ and $\hat{H}^1(G, M) = \hat{H}^{-1}(G, M) = \dfrac{\mathrm{Ker} g}{\mathrm{Im} f}$ (here we use the 2-periodicity of the cohomology groups in the finite cyclic case), which implies that by definition $h(M) = q_{(\sigma-1),\mathrm{Nm}_G}(M)$, when G is finite cyclic.

□

Exercise 4.2. Let K be an algebraically closed field, let m be a non-negative integer and let μ_m denote the group of m-th roots of unity in K. Let G be a group of automorphisms of K of order m. Show that the map $H^2(G, \mu_m) \to H^2(G, K^*)$ is an isomorphism.

Proof. Let's consider the following exact sequence of G-modules

$$0 \longrightarrow \mu_m \longrightarrow K^* \xrightarrow{m} K^* \longrightarrow 0 \ ,$$

it's exact since our field is given algebraically closed (otherwise the map m is not surjective). Then applying Group cohomology, we get a long exact sequence and I write here just the terms of our interest

$$\ldots \longrightarrow H^1(G, K^*) \longrightarrow H^2(G, \mu_m) \xrightarrow{\star} H^2(G, K^*) \xrightarrow{m} H^2(G, K^*) \longrightarrow \ldots \ .$$

Now it's not difficult to see that $\star$ is an isomorphism, since by Hilbert's Satz 90 we have that $H^1(G, K^*) = 0$ (where K is considered as a Galois extension over the field of elements in K fixed by G) and by Corollary 1.31 of Chapter 2 in Milne we have that $m(H^2(G, K^*)) = 0$, since G is of cardinality m. □

Exercise 4.3. Let k be a finite field and let $\bar{k}$ be its algebraic closure. Show that

$$H^2(\mathrm{Gal}(\bar{k}/k), \bar{k}^*) = 0$$

Proof. We show that when k is a finite field, for any finite extension l of it, we have $H^2(\mathrm{Gal}(l/k), l^*) = 0$, then taking the direct limit we have the result of the exercise.
We recall that any finite extension of a finite field is cyclic (so we have the 2-periodicity of the Tate cohomology groups) and also that the Herbrand quotient is 1 when the considered G-module is finite (as in our case). By Hilbert's Satz 90, $\hat{H}^1(\mathrm{Gal}(l/k), l^*) = 0$ and so from the Herbrand quotient (and the 2-periodicity) we deduce that $H^2(\mathrm{Gal}(l/k), l^*) = \hat{H}^0(\mathrm{Gal}(l/k), l^*) = 0$. □

Exercise 4.4. Let G be a profinite group and M a finitely generated discrete G-module which is torsion free as an abelian group. Does this imply that $H^1(G, M)$ is finite?

Proof. The answer is YES.
Since M is finitely generated as a G-module with generators $m_1, \ldots, m_n$, we have that the largest subgroup $H \subset G$ which acts trivially on M is open since $H = \bigcap_{i=1,\ldots,n} \mathrm{Stab}(m_i)$ and each $\mathrm{Stab}(m_i)$ is open (look at the definition of discrete G-module in Milne). Then H is of finite index in G (property of profinite group) and M as H-module is a finitely generated torsion free abelian group.
If we consider the Inflation-Restriction sequence (the fact that G is profinite doesn't create any problem, since the Inf-Res sequence holds also for profinite group. Prove it!)

$$0 \longrightarrow H^1(G/H, M) \xrightarrow{Inf} H^1(G, M) \xrightarrow{Res} H^1(H, M),$$

we see that $H^1(G/H, M)$ is finite since G/H is finite and M is finitely generated (Milne Cor. 1.32). So in order to get our result it's sufficient to show that $H^1(H, M) = 0$.
Since M as an H-module is a finitely generated torsion free abelian group, we have $M \cong \mathbb{Z}^r$ for a certain $r \in \mathbb{N}$. Moreover since the action of H on M is trivial, $H^1(H, M) = \mathrm{Hom}_{cts}(H, \mathbb{Z}^r)$. Given $\psi \in \mathrm{Hom}_{cts}(H, \mathbb{Z}^r)$, since H is compact in G (clopen), its image under ψ has to be compact, but $\mathbb{Z}^r$, with the discrete topology, has no compact subsets other than 0. So we finally get that $H^1(H, M) = \mathrm{Hom}_{cts}(H, \mathbb{Z}^r) = 0$, as we wanted to prove. $\square$