Homework set 3: homological algebra, due November 7 2011

Exercise 3.1. Suppose A is an abelian group which is torsion (every element has finite order). Show that $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. Hint: use an injective resolution of $\mathbb{Z}$.

Exercise 3.2. Explicitly list representatives of all isomorphism classes of extensions of the group $\mathbb{Z}/p\mathbb{Z}$ by the group $\mathbb{Z}/p\mathbb{Z}$. Here an exension is an exact sequence of groups of the form $0 \to \mathbb{Z}/p\mathbb{Z} \to X \to \mathbb{Z}/pZ \to 0$, and two of these are said to be isomorphic if there is a morphism of short exact sequences between them, where the map on the outer two $\mathbb{Z}/p\mathbb{Z}$ is the identity map. Hint: show that there are p of them.

Exercise 3.3. On the category of finite abelian groups, consider the bifunctors $\mathrm{Hom}(A, B)$ and $D(A \otimes D(B))$, where $D(A) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ denotes the dual of A. Show that both functors are left-exact in both A and B, contravariant in A and covariant in B. Show also that the two bifunctors are equivalent.

Exercise 3.4. Of the 64 bifunctors mentioned in class, which other ones are also left-exact in both A and B, contravariant in A and covariant in B? Can you show that they are all equivalent? Recall, we get to 64 by considering Hom, $\otimes$, Ext and Tor (factor 4) taken duals of A, B and the result (factor 8) and switching A and B (factor 2).

Exercise 3.5. For an abelian group A let $T(A)$ be the torsion subgroup of A, i.e., the subgroup consisting of all elements of finite order. Show that T is a left exact functor, and compute its derived functors. Hint: $T(A)$ is the kernel of $A \to A \otimes \mathbb{Q}$.

Exercise 3.6. Show that $H^1(G, \mathbb{Z}) = 0$ for any finite group G, with trivial action on $\mathbb{Z}$. If G is cyclic of order 2, and $\mathbb{Z}^-$ is $\mathbb{Z}$ with non-trivial G-action, compute $H^1(G, \mathbb{Z}^-)$.