Topics in field theory: exercises

Mathematisch Instituut, Universiteit Leiden, Fall 2013

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Exercise 1. A field K is called *real closed* if it satisfies the following three conditions: K^{*2} is a subgroup of index 2 of K^*, generated by $-K^{*2}$; each sum of two squares in K is a square in K; and each polynomial in $K[X]$ of odd degree has a zero in K.

Prove that each real closed field has characteristic 0.

Exercise 2. Let K be a field. Suppose that for each $a \in K$ there exist $n \in \mathbf{Z}_{>0}$ and $x_1, \ldots, x_n \in K$ such that $a = \sum_{i=1}^n x_i^2$. Prove that there exists $n \in \mathbf{Z}_{>0}$ such that for each $a \in K$ there exist $x_1, \ldots, x_n \in K$ such that $a = \sum_{i=1}^n x_i^2$.

Exercise 3. Let K be a finite field.

(a) Prove that there exist a, $b \in K$ with $a^2 + b^2 = -1$.

(b) Let $n \in \mathbf{Z}_{>0}$. Show that the subset $\{\sum_{i=1}^n x_i^2 : x_i \in K\}$ of K is closed under multiplication.

Exercise 4. Let X be a discrete topological space. Consider the set X^X with the product topology.

(a) Show that the subset of X^X consisting of the injective maps $X \to X$ is closed.

(b) If X is infinite, show that the set of surjective maps $X \to X$ is dense in X^X.

Exercise 5. Suppose that A and B are discrete groups, i.e., groups with the discrete topology.

(a) Show that $\mathrm{Hom}(A, B)$ is a closed subset of B^A.

(b) If A and B are extension fields of a field K, show that the set of field homomorphisms $A \to B$ whose restriction to K is the identity map, is closed in B^A.

Exercise 6. Let G be a topological group and H a subgroup.

(a) Let $a \in G$. Show that H is open in G if and only if aH is open in G and that H is closed in G if and only if aH is closed in G.

(b) Show that H is closed if H is open. Show that H is open if H is closed and H is of finite index in G.

(c) Assume G is compact. Show that H is open if and only if H is closed and H has finite index in G.

Exercise 7. Prove that every continuous bijection from one profinite group to another is a homeomorphism.

Exercise 8. Let L be the field obtained from $\mathbf{Q}$ by adjoining all $a \in \overline{\mathbf{Q}}$ with $a^2 \in \mathbf{Q}$. Prove: L is Galois over $\mathbf{Q}$, and $\mathrm{Gal}(L/\mathbf{Q})$ is isomorphic to the product of a countably infinite number of copies of $\mathbf{Z}/2\mathbf{Z}$.

Exercise 9. Show that every infinite profinite group is uncountable.