Topics in field theory: exercises
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Exercise 97. Let k be a field of characteristic $p > 0$. Show that k has a perfect closure k^{pf}, i.e., an extension field of k, which is itself perfect, and which can be embedded as an extension of k in every perfect field extension of k. Show that unlike the algebraic closure, the perfect closure is *uniquely* unique.

Exercise 98 (preferred). Let L and M be two intermediate fields of an algebraic field extension $K \subset \Omega$.
(a) Show that $L \cap M = K$ if L and M are linearly disjoint over K.
(b) Show that the converse of (a) does not hold.
(c) Show that the converse of (a) does hold when L/K is a Galois extension.

Exercise 99. Suppose we have subfields K, L, M, N of a field Ω for which $K \subset L$ and $K \subset M \subset N$. Show that the following are equivalent:
(1) L and N are linearly disjoint over K
(2) L and M are linearly disjoint over K and LM and N are linearly disjoint over M.

Exercise 100 (preferred). Suppose that $K \subset L$ is a field extension for which K is algebraically closed in L. Suppose also that $K \subset M$ is a primitive field extension, i.e., M can be generated by a single element as a field extension of K. Show that L and M are linearly disjoint.

Exercise 101 (preferred). Does the statement in the previous exercise hold without the condition that M is primitive over K?

Exercise 102. Let $k \subset K$ be a field extension, and let $\mathbf{F}$ be the prime field of k. Show that the following are equivalent:
(1) the natural map $\Omega_{k/\mathbf{F}} \otimes_k K \to \Omega_{K/\mathbf{F}}$ is injective;
(2) for every vector space M over K, every k-derivation $k \to M$ can be extended to a K-derivation $K \to M$.

Exercise 103 (preferred). Let k be a field and let K be a field extension of k generated by elements $a_1, \ldots, a_n \in K$. Show that K is a separable algebraic extension if and only if there are polynomials $f_1\ \ldots, f_n \in k[x_1, \ldots, x_n]$ such that $f_i(a_1, \ldots, a_n) = 0$ for $i = 1, \ldots, n$ and

$$\det(\frac{\partial f_i}{\partial x_j}(a_1, \ldots, a_n)) \neq 0.$$

Exercise 104. Let k be a field, let $k(x)$ be field of rational funtions in a single variable, and let $k(x, y)$ be the extension of $k(x)$ given by $x^2 + y^2 = 1$. Show that $k(x, y)$ is a purely transcendental extension of k.

Exercise 105 (preferred). Suppose that $K \subset L$ is an algebraic extension, and suppose that $M = K(x_1, \ldots, x_n)$ is the field of fractions of the polynomial ring $K[x_1, \ldots, x_n]$. Show that $L \otimes_K M$ is a field. Is the same true if we do not assume that $K \subset L$ is algebraic?

Exercise 106. Suppose that $K \subset L$ is a field extension such that L is algebraically closed and L has finite transcendence degree over K. Show that every K-algebra homomorphism $L \to L$ is a field automorphism.