Topics in field theory: exercises
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Exercise 107. Let V be a vector space over a field K, and let $V^* = \mathrm{Hom}_K(V, K)$ be its dual.
(a) Show that there is a linear map $f\colon\ V \otimes_K V^* \to \mathrm{End}(V)$ sending $v \otimes \phi$ to the map $w \mapsto \phi(w)v$.
(b) Show that f is an isomorphism if the dimension of V is finite.
(c) For finite dimensional V show that the composition

$$\mathrm{End}(V) \to V \otimes V^* \to K$$

of the map f^{-1} and the map given by $v \otimes \phi \mapsto \phi(v)$ is equal to the trace map.
(d) When V is not finite dimensional, show that f is injective and describe its image.

Exercise 108. Let R be ring and M a left R-module. Show that M has a unique maximal semisimple submodule. This module is called the *socle* of M. For an integer n, what is the socle of $\mathbf{Z}/n\mathbf{Z}$ as a $\mathbf{Z}$-module?

Exercise 109. Is it true that for every ring R and every sequence of simple R-modules $M_1, M_2, \ldots$ the product $\prod_i M_i$ is semisimple?

Exercise 110 (preferred). Let K be a field, and consider the ring $\mathrm{End}(K)$ of endomorphisms of the additive group of K with product topology $\mathrm{End}(K) \subset \mathrm{Map}(K, K) = \prod_{e \in K} K$. View K is a subring of $\mathrm{End}(K)$ by letting K act on K as left multiplications. In particular $\mathrm{End}(K)$ is a left K-module. Show that every finitely generated left K-submodule of $\mathrm{End}(K)$ is closed.

Exercise 111. Let V be a vector space over K, and let $R \subset \mathrm{End}_K(V)$ be a subring with the property that for any two independent $v, v' \in V$ and any $w, w' \in V$ there is an element $a \in R$ with $av = w$ and $a'v' = w'$. Show that R is dense in $\mathrm{End}_K(V)$.

Exercise 112 (preferred). Let G be a finite group of automorphisms of a field L. Consider L as a subring of $\mathrm{End}(L)$, and let LG be the subring generated by L and $G \subset \mathrm{End}(L)$. Without using the Jacobson-Bourbaki correspondence show that every intermediate ring of the extension $L \subset LG$ is equal to LH for a unique subgroup H of G.

Exercise 113. Suppose $K \subset L$ is a finite field extension. Let $x_1, \ldots, x_n$ be a K-basis of L and let $f_1, \ldots, f_n$ be an L-basis of $\mathrm{End}_K(L)$. Show that the $n \times n$-matrix $(f_i(x_j))_{i,j}$ over L is invertible.

Exercise 114 (preferred). Let p be a prime number and let k be the field of p elements. Let L be the field $k(x)$ where x is transcendental over k, let $n \geq 0$ and let $K = k(x^{p^n})$.
(a) For $i = 0, \ldots, p^n - 1$ show that there is an element $D_i \in \text{End}_K(L)$ with $D_i(x^j) = \binom{j}{i} x^{j-i}$ for all $j \in \mathbf{Z}$.
(b) Show that

$$D_j(ab) = \sum_{i=0}^{j} D_{j-i}(a) D_i(b)$$

when $0 \leq j < p^n$ and $a, b \in L$.
(c) Show that $D_i D_j = 0$ if $i + j \geq p^n$ and $D_i D_j = \binom{i+j}{i} D_{i+j}$ otherwise.
(d) Show that $D_0, D_1, \ldots, D_{p^n - 1}$ is a basis for $\text{End}_K(L)$ over L.

Exercise 115 (preferred). Let k be a field whose characteristic is not 3. Let $k(x)$ be the field of rational functions in a single variable, and let $k(x, y)$ be the extension of $k(x)$ given by $x^3 + y^3 = 1$. Show that $k(x, y)$ is not a purely transcendental extension of k (cf. Exercise 104).

Exercise 116. Suppose we have field extensions $K \subset L \subset M$ where M is a finitely generated field extension of K. Show that L is a finitely generated field extension of K.

Exercise 117 (preferred). Let K be a perfect field. Show that every algebraic extension of K is perfect.