Topics in field theory: exercises

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A field k is called *real closed* if it satisfies the following three conditions:

(1) each $f \in k[X]$ of odd degree has a zero in k;

(2) the group k^*/k^{*2} has order 2, with generator $-k^{*2}$;

(3) for any a, $b \in k$ there exists $c \in k$ with $a^2 + b^2 = c^2$.

Exercise 118 (preferred). (a) Let $\bar{\mathbf{Q}}$ be an algebraic closure of $\mathbf{Q}$. Prove that $\bar{\mathbf{Q}}$ has a subfield k with $\sqrt{-2} \in k$ and $\sqrt{-1} \notin k$ such that for every field extension $l \neq k$ of k in $\bar{\mathbf{Q}}$ one has $\sqrt{-1} \in l$.

(b) Let k be as in (a). Prove that k is not real closed, but that it does satisfy conditions (1) and (2) above.

Exercise 119. Let p be a prime number, n a positive integer, k a field, $\zeta \in k^*$ an element of order p, and l a Galois extension of k with a cyclic Galois group of order p^n. Prove: l is contained in some Galois extension m of k with $\mathrm{Gal}(m/k)$ cyclic of order p^{n+1} if and only if there exists $\alpha \in l^*$ with $N_{l/k}(\alpha) = \zeta$. (In the case $p^n = 2$, this was proved in class.)

Exercise 120 (preferred). (a) Let k be a real closed field. Prove that the group of roots of unity in k^* has order 2.

(b) Let Ω be an algebraically closed field, and let $\sigma \in \mathrm{Aut}\,\Omega$ be of order 2. Prove that for each root of unity $\zeta \in \Omega^*$ one has $\sigma(\zeta) = \zeta^{-1}$.

Exercise 121. Let k be a field, let $\bar{k}$ be a separable closure of k, and put $G = \mathrm{Gal}(\bar{k}/k)$. Let $H \subset G$ be the closure of the subgroup of G generated by $\{\sigma \in G : \sigma^2 = 1\}$. Prove that $\bar{k}^H$ is the intersection of all real closed subfields $l \subset \bar{k}$ that contain k. (This intersection is defined to be $\bar{k}$ if no such l exist.)

Exercise 122. Let G be a profinite group, and let $\gamma \in G$. The *order* of γ is defined to be the unique Steinitz number s such that the closure of the subgroup generated by γ is isomorphic to $\hat{\mathbf{Z}}/s\hat{\mathbf{Z}}$ (cf. Exercise 95).

Suppose that G is the Galois group of a Galois extension $k \subset l$ of fields, and let $\gamma \in G$. Prove that all cycle lengths of γ in its action on l are finite, and that the least common multiple of all these cycle lengths equals the order of γ.

Exercise 123. A group G is called *dihedral* if there are σ, $\tau \in G$ of order 2 such that $G = \langle \sigma, \tau \rangle$. Prove that for every $n \in \mathbf{Z}_{>0}$ there is, up to isomorphism, exactly one dihedral group of order $2n$, and that there is, up to isomorphism, exactly one infinite dihedral group.

Exercise 124 (preferred). A *prodihedral* group is a profinite group D that contains elements σ, τ of order 2 such that D is the closure of the subgroup generated by σ and τ; in that situation we define s_D to be the order of $\sigma\tau$ (as defined in Exercise 122).

(a) Prove that s_D is well-defined in the sense that it depends only on D and not on the choice of σ and τ.

(b) Prove that there is a bijection from the set of isomorphism classes of prodihedral groups to the set $\mathcal{S}$ of Steinitz numbers that maps the isomorphism class of D to s_D.

(c) With D, σ, τ, s_D as above, prove that σ and τ are conjugate in D if and only if s_D is not divisible by 2.

A field k is called *Pythagorean* if for any a, $b \in k$ there exists $c \in k$ with $a^2 + b^2 = c^2$. (This is condition (3) above.)

Exercise 125. (a) Prove that every field of characteristic 2 is Pythagorean.

(b) Let k be a field with $\operatorname{char} k > 2$. Prove: k is Pythagorean if and only if every element of k is a square in k.

Let k be a field with $\operatorname{char} k \neq 2$, and let $S \subset k$ be defined by $S = \{\sum_{i=1}^{n} x_i^2 : n \in \mathbf{Z}_{>0}, x_i \in k^* \ (1 \leq i \leq n)\}$. It is a theorem, which may be proved in class, that S is the set of elements of k that are positive under every ordering of k. You may assume this theorem in the following exercise.

Exercise 126 (preferred). Let k be a field with $\operatorname{char} k \neq 2$, let $\bar{k}$ be a separable closure of k, and put $G = \operatorname{Gal}(\bar{k}/k)$. Write $G[2] = \{\sigma \in G : \sigma^2 = 1\}$ and $G^2 = \{\sigma^2 : \sigma \in G\}$.

(a) With S as defined above, prove that for $x \in k^*$ one has $x \in S$ if and only if for all $\sigma \in G[2]$ one has $\sigma(\sqrt{x}) = \sqrt{x}$.

(b) Prove: k is Pythagorean if and only if the closure of the subgroup of G generated by $G[2] \cup G^2$ equals G itself.

Exercise 127 (preferred). Let k be a field, let $\bar{k}$ be a separable closure of k, and put $G = \operatorname{Gal}(\bar{k}/k)$.

(a) Suppose that $D \subset G$ is a closed subgroup that is prodihedral. Prove that there is a set T of prime numbers such that the Steinitz number s_D defined in Exercise 124 is equal to $\prod_{l \in T} l^{\infty}$.

(b) Suppose that the number of orderings of k is greater than 1. Prove that G has a closed prodihedral subgroup D with $s_D = 2^{\infty}$.

Exercise 128 (preferred). Prove that there exists a field k such that the absolute Galois group of k is a prodihedral group D with $s_D = 2^{\infty}$, with s_D as in Exercise 124. (You may use the result of Exercise 127(b).) Prove also that any such k satisfies conditions (1) and (3) stated above, but not condition (2).