Topics in field theory: exercises

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Exercise 10.

(a) Let g be an integer, $g > 1$, and define $\mathbf{Z}_g = \varprojlim \mathbf{Z}/g^n\mathbf{Z}$. Prove that $\mathbf{Z}_g$ is, as a profinite ring, isomorphic to $\prod_{p|g} \mathbf{Z}_p$, the product ranging over the primes p dividing g.

(b) Define $\hat{\mathbf{Z}} = \varprojlim \mathbf{Z}/n\mathbf{Z}$, the limit ranging over the set of positive integers n, ordered by divisibility. Prove that $\hat{\mathbf{Z}}$ is isomorphic as a profinite ring to $\prod_p \mathbf{Z}_p$, the product ranging over all primes p.

Exercise 11. Let p be a prime number, and $\mathbf{Z}_p$ the ring of p-adic integers defined above. Prove:

(a) $\mathbf{Z}_p^* = \mathbf{Z}_p \backslash p\mathbf{Z}_p$;

(b) each $a \in \mathbf{Z}_p \backslash \{0\}$ can be uniquely written in the form $a = up^n$ with $u \in \mathbf{Z}_p^*$ and $n \in \mathbf{Z}$, $n \geq 0$;

(c) the ring $\mathbf{Z}_p$ is a local domain with residue class field $\mathbf{F}_p$.

Exercise 12 (preferred).

(a) Prove that each $a \in \hat{\mathbf{Z}}$ has a unique representation as $a = \sum_{n=1}^{\infty} c_n n!$ with $c_n \in \mathbf{Z}$, $0 \leq c_n \leq n$.

(b) Let b be a non-negative integer, and define the sequence $(a_n)_{n=0}^{\infty}$ of non-negative integers by $a_0 = b$ and $a_{n+1} = 2^{a_n}$. Prove that $(a_n)_{n=0}^{\infty}$ converges in $\hat{\mathbf{Z}}$, and that the limit is independent of the choice of b.

(c) Let $a = \lim_{n\to\infty} a_n \in \hat{\mathbf{Z}}$ be as in (b), and write $a = \sum c_n n!$ as in (a). Determine c_n for $1 \leq n \leq 10$.

Exercise 13 (preferred). Prove that there is an isomorphism of profinite groups $\hat{\mathbf{Z}}^* \cong \hat{\mathbf{Z}} \times \prod_{n=1}^{\infty} \mathbf{Z}/n\mathbf{Z}$.

Exercise 14 (preferred). Let p be a prime number, and view $\mathbf{Z}_p = \varprojlim \mathbf{Z}/p^n\mathbf{Z}$ as a closed subgroup of the profinite group $A = \prod_{n=1}^{\infty} \mathbf{Z}/p^n\mathbf{Z}$. Prove that A and $\mathbf{Z}_p \times (A/\mathbf{Z}_p)$ are isomorphic as groups but not as profinite groups.

Exercise 15. Suppose that for each i in some set I we are given a profinite group G_i. Show that $\prod_{i\in I} G_i$ is a profinite group.

Exercise 16. The *profinite completion* of a group G is the profinite group $\hat{G}$ defined by $\hat{G} = \varprojlim G/N$, with N ranging over the set of normal subgroups of G of finite index, ordered by containment, the transition maps being the natural ones.
(a) Prove that there is a natural group homomorphism $G \to \hat{G}$, and that its image is dense in $\hat{G}$.
(b) Find a group G for which this homomorphism is not injective.

Exercise 17 (preferred).
(a) Show that every subgroup of $\hat{\mathbf{Z}}$ of finite index is open in $\hat{\mathbf{Z}}$.
(b) Is the same true for the profinite group $G = C_2 \times C_2 \times \cdots$, a countably infinite product of groups of order 2?
(c) Is there a group A such that this group G is the profinite completion (see previous exercise) of A?

Exercise 18 (preferred).
(a) Show that $\mathrm{Hom}(\mathbf{Q}^*, \{\pm 1\})$ is a closed subgroup of $\{\pm 1\}^{\mathbf{Q}^*}$, and that there is an isomorphism of topological groups $\mathrm{Hom}(\mathbf{Q}^*, \{\pm 1\}) \cong \{\pm 1\}^{\mathcal{P}}$ where $\mathcal{P} = \{-1\} \cup \{p : p \text{ is prime}\}$.
(b) Suppose that $G \subset \mathrm{Hom}(\mathbf{Q}^*, \{\pm 1\})$ is a closed subgroup such that for all $a \in \mathbf{Q}^*$ with $a \notin \mathbf{Q}^{*2}$ there is $f \in G$ with $f(a) = -1$. Show that $G = \mathrm{Hom}(\mathbf{Q}^*, \{\pm 1\})$.
(c) Show that $\mathrm{Gal}(\mathbf{Q}(\sqrt{\mathbf{Q}^*})/\mathbf{Q}) \cong \mathrm{Hom}(\mathbf{Q}^*, \{\pm 1\})$.

Exercise 19. Let K be a field, with separable closure K^{sep}. Show that there is a unique maximal abelian extension K^{ab} of K inside K^{sep}. Write $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Prove that K^{ab} is a Galois extension of K, and that $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is isomorphic to $G_K/[G_K, G_K]$, where $[G_K, G_K]$ denotes the closure of the commutator subgroup of G_K.