Topics in field theory: exercises
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Exercise 20. Let $K \subset L$ be a Galois extension of fields, $S \subset \mathrm{Gal}(L/K)$ any subset, and $E = \{x \in L : \sigma(x) = x \text{ for all } \sigma \in S\}$. Prove that $\mathrm{Gal}(L/E)$ is the closure of the subgroup of $\mathrm{Gal}(L/K)$ generated by S.

Exercise 21 (preferred).
(a) Let E be a torsion abelian group. Prove that E has exactly one $\hat{\mathbf{Z}}$-module structure, and that the scalar multiplication $\hat{\mathbf{Z}} \times E \to E$ defining this module structure is continuous, if E is given the discrete topology.
(b) For $E = \mathbf{Q}/\mathbf{Z}$ prove that the map $\hat{\mathbf{Z}}^* \to \mathrm{Aut}(E)$ induced by (a) is an isomorphism of groups.

Exercise 22. Let G be a profinite group acting on a set E with the discrete topology. Prove that the action is continuous if and only if for each $e \in E$ the stabilizer $G_e = \{g \in G : ge = e\}$ is open in G, and for finite E if and only if the kernel $\{g \in G : ge = e \text{ for all } e \in E\}$ of the action is open in G.

Exercise 23 (preferred). Let G be a group with profinite completion $\hat{G}$. Prove that the category of finite sets with a an action of G is equivalent to the category of finite discrete sets witha a continuous action of $\hat{G}$.

Exercise 24 (preferred). Show that every profinite group is isomorphic to the Galois group of some field extension.

Exercise 25. Show that the field of complex numbers has infinitely many automorphisms.

Exercise 26 (preferred). Let $K \supset k$ be a finitely generated field extension, and let $K \supset E \supset k$ be a subextension. Show that E is a finitely generated field extension of k.

Exercise 27. Let K be a field of characteristic $p > 0$, and let $\alpha \in K$ be an element which has no p-th root in K. Show that for every $n \geq 0$ the polynomial $X^{p^n} - \alpha \in K[X]$ is irreducible.

Exercise 28. Let $F \subset E$ be an algebraic field extension. Show that every subring of E which contains F is a field. Is this necessarily true if E is not algebraic over F? Prove or give a counterexample.

Exercise 29 (preferred). Let k be a field of characteristic $p > 0$ and let t, u be elements of an extension field of k that are algebraically independent over k. Prove the following:
(a) $k(t, u)$ has degree p^2 over $k(t^p, u^p)$.
(b) There exist infinitely many extensions between $k(t, u)$ and $k(t^p, u^p)$.