Topics in field theory: exercises
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http://www.math.leidenuniv.nl/~desmit/edu/topics_ft_2013/

Exercise 30. Let k be a field. We say that k is *separably closed* if the only finite separable field extension of k is k itself. We say that k has a *uniquely unique* algebraic closure if for any two algebraic closures $\bar{k}$ and $\tilde{k}$ of k there is a unique field isomorphism $\bar{k} \to \tilde{k}$ that is the identity on k. Prove: k is separably closed if and only if k has a uniquely unique algebraic closure.

Note. In Exercises 31–38, all tensor products are taken over $\mathbf{Z}$.

Exercise 31. Let A and B be additively written abelian groups. Assume that A is *divisible*, which means that for each $a \in A$ and each $n \in \mathbf{Z}_{>0}$ there exists $b \in A$ with $a = nb$. Assume also that B is *torsion*, which means that each element of B has finite order. Prove: $A \otimes B = 0$.

Exercise 32 (preferred). Describe the group $A \otimes B$ when each of A and B is one of the following: (a) finite cyclic; (b) infinite cyclic; (c) the Klein four group; (d) the additive group of a vector space over $\mathbf{Q}$; and (e) $\mathbf{Q}/\mathbf{Z}$. (Be sure to cover all combinations.)

Exercise 33. Is the tensor product of two finitely generated abelian groups finitely generated? Is the tensor product of two finite abelian groups finite? Is the tensor product of a finitely generated abelian group and a finite abelian group finite? Give in each case a proof or a counterexample.

Exercise 34 (preferred). Suppose that A and B are non-zero finitely generated abelian groups. Prove: $A \otimes B = 0$ if and only if A and B are finite with $\gcd(\#A, \#B) = 1$.

Exercise 35. (a) Give an example of abelian groups A, B, C and a $\mathbf{Z}$-bilinear map $f: A \times B \to C$ such that the image of f is not a subgroup of C.

(b) Give an example of abelian groups A, B such that not every element of $A \otimes B$ is of the form $a \otimes b$, with $a \in A$, $b \in B$.

Note. Rings are always supposed to have a unit element, ring homomorphisms are required to preserve the unit element, and subrings are required to contain the unit element.

Exercise 36. Let R be the ring $\prod_p \mathbf{Z}/p\mathbf{Z}$, where p ranges over the set of prime numbers, and where the ring operations are componentwise. Prove that $\mathbf{Z}$ may be considered as a subring of R, that $A = \{r \in R : \text{ there exists } n \in \mathbf{Z}_{>0} \text{ with } nr \in \mathbf{Z}\}$ is a subring of R, that $I = \bigoplus_p \mathbf{Z}/p\mathbf{Z}$ is an ideal of R contained in A, and that there is a ring isomorphism $A/I \cong \mathbf{Q}$.

Exercise 37 (preferred). Let the ring A be as in Exercise 36. Prove that there is a unique group homomorphism $A \otimes A \to A$ that for each a, $b \in A$ sends $a \otimes b$ to ab, and that this group homomorphism is an *isomorphism.* (You may assume the results of Exercise 36.)

Exercise 38. Suppose that A and B are commutative rings. Prove that the following two conditions are equivalent:

(i) there is a commutative ring C that contains subrings A' and B' with $A' \cong A$ and $B' \cong B$;

(ii) the maps $A \to A \otimes B$, $a \mapsto a \otimes 1$ and $B \to A \otimes B$, $b \mapsto 1 \otimes b$ are injective.

Notation. Let R be a ring. The *characteristic* $\operatorname{char} R$ of a ring R is defined to be the non-negative integer that generates the kernel of the unique ring homomorphism $\mathbf{Z} \to R$.

Exercise 39 (preferred). Suppose that A and B are commutative rings.

(a) Prove: if condition (i) of Exercise 38 is satisfied, then one has $\operatorname{char} A = \operatorname{char} B$.

(b) Prove: if $\operatorname{char} A = \operatorname{char} B \neq 0$, then condition (i) of Exercise 38 is satisfied.

Notation. For an additively written abelian group A and $n \in \mathbf{Z}$, we write nA and $A[n]$ for the image and for the kernel, respectively, of the map $A \to A$, $x \mapsto nx$.

Exercise 40 (preferred). (a) Give an example of commutative rings A and B with $\operatorname{char} A = \operatorname{char} B = 0$ such that condition (i) of Exercise 38 is not satisfied.

(b) Suppose that A and B are commutative rings with $\operatorname{char} A = \operatorname{char} B = 0$. Prove that condition (i) of Exercise 38 is equivalent to the following condition: if $n \in \mathbf{Z}$ is such that $nA = A$ then $B[n] = 0$, and if $n \in \mathbf{Z}$ is such that $nB = B$ then $A[n] = 0$.

Note added later. Part (b) of this Exercise is incorrect: the condition stated is necessary but not sufficient. There will hopefully later be a corrected exercise.

Exercise 41. (a) Give an example of a ring R, a right R-module N, and an exact sequence $0 \to L \to M$ of left R-modules, such that the induced sequence $0 \to N \otimes_R L \to N \otimes_R M$ is not exact.

(b) Let k be a field, let N be a k-vector space, and let $K \to L \to M$ be an exact sequence of k-vector spaces. Prove that the induced sequence $N \otimes_k K \to N \otimes_k L \to N \otimes_k M$ is exact.