Topics in field theory: exercises

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Exercise 42. (a) Let $k \subset l$ be a Galois extension of fields, with group G, let W be an l-vector space, and suppose that a semilinear action of G on W is given. Prove that the action is continuous if and only if for each $w \in W$ the stabilizer $G_w = \{\sigma \in G : \sigma w = w\}$ of w in G is an *open* subgroup of G.

(b) Give an example of k, l, G, W as in (a), with a semilinear action of G on W that is *not* continuous.

Exercise 43. Let k, l, G be as in Exercise 42(a), and let $U \to V \to W$ be an exact sequence of l-vector spaces. Suppose that each of U, V, W is equipped with a continuous semilinear action of G that is respected by the maps in the sequence. Prove that a sequence $U^G \to V^G \to W^G$ of k-vector spaces is induced, and that this sequence is exact.

Exercise 44. Let $k \subset l$ be any field extension, and let W be an l-vector space. By a *k-form of W* we mean a k-subspace $V \subset W$ with the property that the map $V \otimes_k l \to W$, $v \otimes c \mapsto cv$, is an isomorphism of l-vector spaces.

(a) Prove that there is a k-form of W.

(b) Suppose that l is Galois over k, with group G. Establish a bijection between the set of k-forms of W and the set of semilinear continuous actions of G on W.

Exercise 45 (preferred). Let k, l, G be as in Exercise 42(a), and let $n \in \mathbf{Z}_{\geq 0}$. We give the group $\mathrm{Gl}(n,l)$ of invertible $n \times n$-matrices over l the discrete topology, and we let G act entry-wise on $\mathrm{Gl}(n,l)$. Let $c\colon G \to \mathrm{Gl}(n,l)$ be a function. Prove that the following two conditions are equivalent:

(i) c is continuous, and for all σ, $\tau \in G$ one has $c(\sigma\tau) = c(\sigma) \cdot \sigma(c(\tau))$;

(ii) there exists $b \in \mathrm{Gl}(n,l)$ such that for all $\sigma \in G$ one has $c(\sigma) = b \cdot (\sigma b)^{-1}$.

Exercise 46. Let A be an abelian group and F be a field. The *lemma of Dedekind-Artin* asserts that any finite set S of group homomorphisms $A \to F^*$ is linearly independent over F in the sense that the only vector $(c_\sigma)_{\sigma \in S}$ with entries c_σ in F with the property that each $a \in A$ satisfies $\sum_{\sigma \in S} c_\sigma \sigma(a) = 0$, is the zero vector.

Show that this lemma can be deduced from the Chinese remainder theorem. (*Hint.* First extend each $\sigma \in S$ to a ring homomorphism from the group ring $F[A]$ to F.)

Note. See the Leiden Algebra III notes, 23.15 and Opgave 6, for a direct proof of the Lemma of Dedekind-Artin.

Exercise 47 (preferred). Let $k \subset l$ be a Galois extension of fields with a finite cyclic Galois group G, and let γ be a generator of G. Denote by $\mathrm{N}\colon l \to k$ the norm map, defined by $\mathrm{N}(x) = \prod_{\sigma \in G} \sigma(x)$, for $x \in l$. Let $c \in l^*$. Prove that the following statements are equivalent:

(i) there exists a 1-cocycle $a\colon G \to l^*$ with $a(\gamma) = c$;

(ii) $\mathrm{N}(c) = 1$;

(iii) there exists $b \in l^*$ with $c = b/\gamma(b)$.

Exercise 48 (preferred). The *additive Hilbert Theorem 90* asserts that, for a Galois extension $k \subset l$ of fields with group G and a continuous function $a\colon G \to l$, the following two statements are equivalent:

(i) the map a is continuous if l is given the discrete topology, and for all σ, $\tau \in G$ one has $a(\sigma\tau) = a(\sigma) + \sigma(a(\tau))$;

(ii) there exists $b \in l$ such that for all $\sigma \in G$ one has $a(\sigma) = b - \sigma(b)$.

In class, a proof of this theorem was sketched. Provide the details of this proof.

Exercise 49. Formulate and do the additive analogue of Exercise 47. (You may assume the additive Hilbert Theorem 90.)

Exercise 50. The *normal basis theorem*, which will be proved in class, asserts that for every finite Galois extension $k \subset l$ of fields there exists $\alpha \in l$ such that the elements $\sigma\alpha$, with $\sigma \in \mathrm{Gal}(l/k)$, form a basis of l as a vector space over k. Such a basis is called a *normal basis* of l over k.

Deduce the additive Hilbert Theorem 90 (see Exercise 48) from the normal basis theorem.

Exercise 51 (preferred). Let $k \subset l$ be a Galois extension of fields of characteristic different from 2, and suppose that $\mathrm{Gal}(l/k)$ is a non-cyclic group of order 4.

(a) Prove that there are α, $\beta \in l$ such that $\alpha^2 \in k$ and $\beta^2 \in k$, and such that 1, α, β, $\alpha\beta$ form a basis of l as a vector space over k.

(b) Let α, β be as in (a), and let a, b, c, $d \in k$. Prove: $a + b\alpha + c\beta + d\alpha\beta$ belongs to a normal basis of l over k (see Exercise 50) if and only if all of a, b, c, d are non-zero.

Exercise 52 (preferred). Let p be a prime number, and let $k \subset l$ be a Galois extension of fields of characteristic p with a Galois group G of order p. Denote by γ a generator of G, and by $\mathrm{Tr}\colon l \to k$ the trace map, defined by $\mathrm{Tr}(x) = \sum_{\sigma \in G} \sigma(x)$, for $x \in l$. Let $\gamma - 1\colon l \to l$ be the map sending $x \in l$ to $\gamma(x) - x$.

(a) Prove: the pth iterate $(\gamma - 1)^p$ is the zero map, and $(\gamma - 1)^{p-1} = \mathrm{Tr}$.

(b) Let $\alpha \in l$. Prove that α belongs to a normal basis of l over k (see Exercise 50) if and only if $\mathrm{Tr}(\alpha) \neq 0$.