Topics in field theory: exercises

Mathematisch Instituut, Universiteit Leiden, Fall 2013
Hendrik Lenstra & Bart de Smit

Exercise 53. Let G be a group of order 2, and let A be the group $\mathbf{Z}/4\mathbf{Z}$. Prove that the number of G-module structures on A equals 2, and compute the groups $Z^1(G, A)$, $B^1(G, A)$ and $H^1(G, A)$ for both of them.

Exercise 54. Let G be a group and let $\mathbf{Z}[G]$ be the group ring of G over $\mathbf{Z}$. Let $I(G) \subset \mathbf{Z}[G]$ be the kernel of the ring homomorphism $\mathbf{Z}[G] \to \mathbf{Z}$ that sends all elements of G to 1. For any G-module A give an isomorphism $Z^1(G, A) \to \mathrm{Hom}_G(I(G), A)$. Can you describe the image of $B^1(G, A)$?

Exercise 55 (preferred). Let G be a group and let A be an abelian group. Show that we have a G-module structure on $\mathrm{Map}(G, A)$ defined by $(gf)(h) = f(hg)$ for $f \in \mathrm{Map}(G, a)$ and $g, h \in G$. For $c \in Z^1(G, \mathrm{Map}(G, A))$ define a: $G \to A$ by $a(g) = c(g)(1)$. Show that $ga - a = c(g)$ for all $g \in G$. Deduce that $H^1(G, \mathrm{Map}(G, A)) = 0$.

Exercise 56. Let G be a finite cyclic group generated by σ, and let A be a G-module. Consider the norm map N: $A \to A$ given by $a \mapsto \sum_{\tau \in G} \tau a$, and the map f: $A \to A$ given by $a \mapsto \sigma a - a$. Show that $H^1(G, A) \cong \ker N / \operatorname{im} f$.

Exercise 57. Suppose that G is a finite group of order n, and that A is a G-module. For $c \in Z^1(G, A)$ let $a = \sum_{g \in G} c(g) \in A$. Show that $(g - 1)a = -nc(g)$. Deduce that $H^1(G, A)$ is annihilated by n.

Exercise 58 (preferred). Let G be a finite group, and let A be a G-module which is finitely generated as an abelian group. Show that $H^1(G, A)$ is finite. (You may use Ex. 57.)

Exercise 59 (preferred). For a locally compact and Hausdorff topological abelian group G consider the dual group $\widehat{G} = \mathrm{CHom}(G, \mathbf{R}/\mathbf{Z})$ with the compact-open topology (subbasic open sets are of the form $\{\chi \in \widehat{G}\colon \chi(K) \subset O\}$ where $K \subset G$ is a compact subset and $O \subset \mathbf{R}/\mathbf{Z}$ is open). You may use the Pontryagin duality theorem, which says that the natural map $G \to \widehat{\widehat{G}}$ is an isomorphism of topological groups.

(a) Show that G is discrete if and only if $\widehat{G}$ is compact.
(b) What is the dual of $\mathbf{Q}/\mathbf{Z}$ (with discrete topology)?
(c) For which G is $\widehat{G}$ profinite?
(d) What is the dual of $\mathbf{R}$ (with euclidean topology)?

Exercise 60 (preferred). Describe the Pontryagin dual of the discrete abelian group $\mathbf{Q}$.

Exercise 61. Let K be a field of characteristic 0. Assume that for every $n \geq 1$ and every finite extension L of K, the index $[L^* : L^{*n}]$ is finite. Show that for each positive integer n, there exist only a finite number of abelian extensions of k of degree n.

Exercise 62. Let $\overline{\mathbf{Q}}$ be a fixed algebraic closure of $\mathbf{Q}$. Let K be a maximal subfield of $\overline{\mathbf{Q}}$ not containing $\sqrt{2}$ (such a subfield exists by Zorn's lemma). Show that every finite extension of K is cyclic.

Exercise 63 (preferred). Compute the intersection of the fields $\mathbf{Q}(\sqrt{42}, \sqrt{60}, \sqrt{110})$ and $\mathbf{Q}(\sqrt{6}, \sqrt{14}, \sqrt{21})$.