Topics in field theory: exercises
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Hendrik Lenstra & Bart de Smit

Exercise 75. Suppose that $K \subset L$ is a separable algebraic extension. Show that every derivation $K \to K$ has an extension to a derivation $L \to L$.

Exercise 76 (preferred). Let A be a commutative ring, and let B and A' be commutative A-algebras. Show that $\Omega_{B/A} \otimes_A A' \cong \Omega_{B \otimes_A A'/A'}$ as modules over $B \otimes_A A'$.

Exercise 77. Let k be an algebraically closed field, and let A be a commutative k-algebra which is finite dimensional as a vector space over k. Assume that A is reduced, i.e., the only nilpotent element in A is 0. Show that as a k-algebra, A is isomorphic to a finite product of copies of k.

Exercise 78. Let A be a commutative ring containing the rational field $\mathbf{Q}$. Suppose that $x \in A$ and $D \in \mathrm{Der}(A, A)$ are such that $Dx = 1$ and $\bigcap_{n=1}^{\infty} x^n A = 0$. Show that x is not a zero-divisor of A.

Exercise 79 (preferred). Let A be a commutative ring, and let $e \in A$ be an idempotent ($e^2 = e$). Show that for any derivation D: $A \to M$ we have $D(e) = 0$. Use this to show that for commutative algebras B, C over A we have $\Omega_{B \times C/A} \cong \Omega_{B/A} \times \Omega_{C/A}$ as modules over $B \times C$.

For a field k and a commutative k-algebra A we say that A is separable over k if for every field extension $k \subset k'$ the ring $A \otimes_k k'$ is reduced (i.e., its only nilpotent element is 0).

Exercise 80. Let k be a field of characteristic $p > 0$, and let $K = k(X)$ be the function field in one variable over k. For $n \geq 1$ let $L_n = K(\sqrt[p^n]{X})$. Is L_n separably generated over k? Is $\bigcup_n L_n$ separably generated over k? Is $\bigcup_n L_n$ separable over k?

Exercise 81 (preferred). Let p be a prime number and let K be a field with $\operatorname{char} K = p$. Write $K^p = \{x^p : x \in K\}$, which is a subfield of K. A subset $S \subset K$ is called *p-independent* if for every finite subset $T \subset S$ one has $[K^p(T) : K^p] = p^{\#T}$. A subset $S \subset K$ is called a *p-basis* of K if S is, under inclusion, a maximal p-independent subset of K. Prove that K has a p-basis, that for every p-basis S of K one has $K = K^p(S)$, and that for any two p-bases S and S' of K one has $\#S = \#S'$.

Below we shall, for a field K of characteristic $p > 0$, denote the cardinality of a p-basis by $\mathrm{i}(K)$, and refer to it as the *imperfection* of K.

Exercise 82. Let K be a field of characteristic $p > 0$. Prove that the dimension $[K : K^p]$ of K as a K^p-vector space equals $p^{\mathrm{i}(K)}$ if $\mathrm{i}(K) < \infty$, and $\mathrm{i}(K)$ otherwise.

Exercise 83. Let $K \subset L$ be an extension of fields of non-zero characteristic.

(a) Prove: if $[L : K] < \infty$ then $\mathrm{i}(L) = \mathrm{i}(K)$ (you may assume the result of Exercise 82).

(b) Prove: if L is algebraic over K, then $\mathrm{i}(L) \leq \mathrm{i}(K)$.

(c) Give an example to show that for L algebraic over K one may have $\mathrm{i}(L) < \mathrm{i}(K)$.

Exercise 84 (preferred). Let $K \subset L$ be an algebraic extension of fields of non-zero characteristic. Write $L_{\mathrm{sep}} = \{\alpha \in L : \alpha \text{ is separable over } K\}$ and $L_{\mathrm{pi}} = \{\alpha \in L : \alpha \text{ is purely inseparable over } K\}$.

(a) Prove: the ring $L_{\mathrm{sep}} \otimes_K L_{\mathrm{pi}}$ is a field, and it is K-isomorphic to a subfield of L.

(b) Prove: if L is normal over K, then $L_{\mathrm{sep}} \otimes_K L_{\mathrm{pi}}$ is K-isomorphic to L.

(c) Prove: if one has $\mathrm{i}(K) = 1$, then $L_{\mathrm{sep}} \otimes_K L_{\mathrm{pi}}$ is K-isomorphic to L.