Topics in field theory: exercises

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http://www.math.leidenuniv.nl/~desmit/edu/topics_ft_2013/

Let M be a module over a ring R. We say that M satisfies the *descending chain condition*, or the *descending chain condition on submodules*, if for every chain $M_0 \supset M_1 \supset M_2 \supset \ldots$ of submodules of M there exists n such that for all $m > n$ one has $M_m = M_n$. We say that M satisfies the *ascending chain condition*, or the *ascending chain condition on submodules*, if for every chain $M_0 \subset M_1 \subset M_2 \subset \ldots$ of submodules of M there exists n such that for all $m > n$ one has $M_m = M_n$. We say that M is *of finite length* if it satisfies both the ascending and the descending chain condition. We say that M is *simple* if the number of submodules of M equals 2. We say that M is *indecomposable* if for any two R-modules L and N with $M \cong L \oplus N$ we have either $L = 0$, $N \neq 0$ or $L \neq 0$, $N = 0$.

Exercise 85. Let M be a module over a ring R. Prove: M is of finite length if and only if there exists, for some $n \in \mathbf{Z}_{\geq 0}$, a chain $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ of submodules of M such that for every $i \in \{0, 1, \ldots, n-1\}$ the R-module M_{i+1}/M_i is simple.

Exercise 86. In class it was shown that each module that satisfies the descending chain condition can be written as the direct sum of finitely many indecomposable modules. Prove that the same is true for modules that satisfy the *ascending* chain condition.

Exercise 87. A ring E is called *local* if the set of non-units of E is an additive subgroup of E. Prove that a ring E is local if and only if the set of non-units is a two-sided ideal of E.

Exercise 88 (preferred). Let k be a field, let $A = k[[x, y]]$ be the ring of power series in two variables x and y over k, and put $\mathfrak{m} = Ax + Ay$.

(a) Prove that A is a commutative local ring with maximal ideal $\mathfrak{m}$.

(b) Prove that for each $n \in \mathbf{Z}_{\geq 0}$ the A-module $\mathfrak{m}^n/\mathfrak{m}^{n+2}$ is an indecomposable A-module of finite length.

Exercise 89 (preferred). Is, for every finite ring R, the number of isomorphism classes of finite indecomposable R-modules finite as well? Give a proof or a counterexample.

Exercise 90. Let G be a group, and let G^2 be the subgroup of G that is generated by $\{\sigma^2 : \sigma \in G\}$.

(a) Prove that G^2 is a normal subgroup of G and that G/G^2 is abelian.

(b) Suppose that G is finite of 2-power order and that G/G^2 has order 2. Prove that G is cyclic.

A field k is called *real closed* if it satisfies the following three conditions:

(1) each $f \in k[X]$ of odd degree has a zero in k;

(2) the group k^*/k^{*2} has order 2, with generator $-k^{*2}$;

(3) for any a, $b \in k$ there exists $c \in k$ with $a^2 + b^2 = c^2$.

Exercise 91. Prove that there are infinitely many pairwise non-isomorphic real closed fields.

Exercise 92 (preferred). Prove that there exists a field that satisfies conditions (2) and (3) above, but that does not satisfy condition (1).

Exercise 93 (preferred). Construct a field satisfying conditions (1) and (3), but not (2). Can you make an example that is not algebraically closed?

A *Steinitz number* is a function from the set of prime numbers to the set $\mathbf{Z}_{\geq 0} \cup \{\infty\}$; a Steinitz number s is usually denoted as $\prod_l l^{s(l)}$, with l ranging over the set of prime numbers (Ernst Steinitz, German mathematician, 1871–1928). Through unique prime factorization, each positive integer is viewed as a Steinitz number. We write $\mathcal{S}$ for the set of Steinitz numbers. For s, $t \in \mathcal{S}$, we say that s *divides* t if for all l one has $s(l) \leq t(l)$. Clearly, each set of Steinitz numbers has a least common multiple and a greatest common divisor in $\mathcal{S}$.

Exercise 94. (a) Prove that each Steinitz number s is the least common multiple of the set of positive integers that divide s.

(b) Let p be a prime number, and let $\bar{\mathbf{F}}_p$ be an algebraic closure of $\mathbf{F}_p$. Prove that there is a bijection from $\mathcal{S}$ to the set of subfields of $\bar{\mathbf{F}}_p$ that sends s to the union of all fields $\mathbf{F}_{p^n}$, with n ranging over the set of positive integers that divide s.

Exercise 95 (preferred). Let $\hat{\mathbf{Z}}$ be as in Exercise 10(b), and for each $s \in \mathcal{S}$, write $s\hat{\mathbf{Z}} = \bigcap n\hat{\mathbf{Z}}$, with n ranging over the set of positive integers that divide s. Prove that there is a bijection $\mathcal{S} \to \{\text{closed subgroups of } \hat{\mathbf{Z}}\}$ sending s to $s\hat{\mathbf{Z}}$.

Exercise 96. (a) Prove that for each profinite group G and each $\gamma \in G$ there is a unique group homomorphism $\hat{\mathbf{Z}} \to G$ that sends 1 to γ, that this group homomorphism is continuous, and that its image is the closure of the subgroup generated by γ.

(b) A profinite group G is called *procyclic* if there exists $\gamma \in G$ such that G is the closure of the subgroup generated by γ. Prove that for each $s \in \mathcal{S}$ the group $\hat{\mathbf{Z}}/s\hat{\mathbf{Z}}$ is procyclic, and that for each procyclic group G there is a unique $s \in \mathcal{S}$ such that $G \cong \hat{\mathbf{Z}}/s\hat{\mathbf{Z}}$.