Topics in group theory: exercises
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Exercise 1. Let $K \subset L$ be a field extension of degree equal to a prime number l different from $\mathrm{char}\, K$, and let $\bar{L}$ be an algebraic closure of L. Prove that there exists $a \in \bar{L}$ such that l does not divide $[K(a) : K]$ and such that there is a field isomorphism $L(a) \xrightarrow{\sim} K(a)[X]/(X^l - a)$ that is the identity on $K(a)$.

Exercise 2. Let G be a finite group acting transitively on a finite set X with $\#X > 1$. Prove that there exists $\sigma \in G$ such that for all $x \in X$ one has $\sigma x \neq x$.

Note. There exists in fact such an element σ with the additional property that σ has prime power order, but this is much more difficult to prove.

Exercise 3 (for students who know algebraic number theory)**.** Assume the difficult statement formulated in Exercise 2. Let $K \subset L$ be a field extension with L a finite extension of $\mathbf{Q}$. Suppose that the norm map $N_{L/K}\colon L^* \to K^*$ is surjective. Prove: $K = L$.

Exercise 4. Let n be a positive integer. In class it was stated that every group of order n is cyclic if and only if n is squarefree and for each prime number p dividing n one has $\gcd(p-1, n) = 1$. Prove the "only if" part of this statement.

Exercise 5. An integer is called *cubefree* if it is not divisible by the cube of a prime number.

Let n be a positive integer. In class it was stated that every group of order n is abelian if and only if n is cubefree, for each prime number p dividing n one has $\gcd(p-1, n) = 1$, and for each prime number p for which p^2 divides n one has $\gcd(p+1, n) = 1$. Prove the "only if" part of this statement.

Exercise 6. A finite group is *nilpotent* if and only if it is isomorphic to the product of finitely many groups of prime power orders.

(a) Prove: each finite abelian group is nilpotent, and each finite nilpotent group is solvable.

(b) Give an example of a finite nilpotent group that is not abelian, and give an example of a finite solvable group that is not nilpotent.

Exercise 7. Let n be a positive integer. In class it was stated that every group of order n is nilpotent if and only if for each prime number p and each positive integer t with p^t dividing n one has $\gcd(p^t - 1, n) = 1$. Prove the "only if" part of this statement.

Exercise 8. A group S is called *simple* if the number of normal subgroups of S equals 2.

Let n be a positive integer. Prove: each group of order n is solvable if and only if for each finite non-abelian simple group S the number n is not divisible by $\#S$.

Exercise 9. Let $1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1$ be a short exact sequence of groups. The sequence is said to *split* if there is a group homomorphism $h\colon A \times C \to B$ such that $h \circ \iota = f$ and $g \circ h = \pi$, where $\iota\colon A \to A \times C$ is the canonical injection and $\pi\colon A \times C \to C$ is the canonical projection. Prove that any such h is automatically an isomorphism, and that the sequence splits if and only if there is a group homomorphism $r\colon B \to A$ with $r \circ f = \mathrm{id}_A$.

Exercise 10. Let $1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1$ be a short exact sequence of groups. The sequence is said to be *semi-split* if there is a group homomorphism h from some semidirect product $A \rtimes C$ to B such that $h \circ \iota = f$ and $g \circ h = \pi$, where $\iota\colon A \to A \rtimes C$ is the canonical injection and $\pi\colon A \rtimes C \to C$ is the canonical projection. Prove that any such h is automatically an isomorphism, and that the sequence is semi-split if and only if there is a group homomorphism $i\colon C \to B$ such that $g \circ i = \mathrm{id}_C$.

Exercise 11. Let G be a group, and let N, M be normal subgroups of G with $N \cap M = \{1\}$. Prove: for any $a \in N$ and $b \in M$ one has $ab = ba$.

Exercise 12. For each group G, let $\varphi_G\colon G \to \operatorname{Aut} G$ be the group homomorphism that sends $a \in G$ to the inner automorphism $x \mapsto axa^{-1}$ of G.

(a) Let G be a group for which φ_G is injective. Prove: $\varphi_{\operatorname{Aut} G}$ is injective.

(b) Suppose that G is a non-abelian simple group. Prove that φ_G is injective and that $\varphi_{\operatorname{Aut} G}$ is an isomorphism.