Topics in group theory: exercises
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Exercise 121. Let G be a group. The lower central series $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \ldots$ of G is, as in Exercise 51, defined by $\Gamma^{(0)} = G$ and $\Gamma^{(i+1)} = [G, \Gamma^{(i)}]$. We put $G^\infty = \bigcap_{i \geq 0} \Gamma^{(i)}$.

Now let N be a normal subgroup of G.

(a) Prove: G/N is nilpotent $\Rightarrow G^\infty \subset N$.

(b) Suppose that G is finite. Prove: $G^\infty \subset N \Rightarrow G/N$ is nilpotent.

Exercise 122. (a) Let n be a positive integer, and let $D_n = C_n \rtimes C_2$, where C_n is cyclic of order n and the non-trivial element of C_2 acts on C_n by $x \mapsto x^{-1}$. Compute the group D_n^∞ defined in Exercise 121.

(b) Construct a group G for which G/G^∞ is not nilpotent.

Exercise 123. Let G be a group, let S be a subnormal subgroup of G (see Exercise 38), and let N be a nilpotent normal subgroup of G. Prove that, with the notation of Exercise 121, one has $(S \cdot N)^\infty = S^\infty$. (In class this was proved for *finite* G.)

Exercise 124. Let n be a positive integer, let H be a Sylow-2-subgroup of $\mathrm{Sym}(2^n)$, and put $S = H \cap \mathrm{Sym}(2^n - 1)$, where $\mathrm{Sym}(2^n - 1)$ is naturally embedded in $\mathrm{Sym}(2^n)$. Prove that S is a subnormal subgroup of H, and that the least non-negative integer t for which there is a chain $S = S_0 \subset S_1 \subset \ldots \subset S_{t-1} \subset S_t = H$ of subgroups with S_i normal in S_{i+1} for each $i < t$, equals n.

Exercise 125. As in Exercise 12, we denote, for each group G, the group homomorphism $G \to \mathrm{Aut}\, G$, $a \mapsto (x \mapsto axa^{-1})$, by φ_G. A group G is called *perfect* if $G = [G, G]$.

Let S be a perfect semisimple group, and let $G \subset \mathrm{Aut}\, S$ be a subgroup with $\varphi_S S \subset G$.

(a) Prove: φ_S is injective, one has $\varphi_S S = \mathrm{Soc}\, G$, and $\mathrm{Aut}\, G$ is isomorphic to the normalizer $N_{\mathrm{Aut}\, S}(G)$ of G in $\mathrm{Aut}\, S$.

(b) Prove: $\varphi_{\mathrm{Aut}\, S}$ is an isomorphism. (Note that for perfect *simple* S this is the content of Exercise 12(b).)

Exercise 126. Let n be a positive integer, let T be a non-abelian simple group, and let S be the direct sum of n copies of T. We write $B = \mathrm{Aut}\, T$ and $A = \mathrm{Aut}\, S$.

(a) Prove that A may be identified with a wreath product $B \wr \mathrm{Sym}(n)$, with respect to the $\mathrm{Sym}(n)$-set $\{1, 2, \ldots, n\}$.

(b) Let H be a subgroup of $\mathrm{Sym}(n)$. Prove that the group $\mathrm{Aut}(B \wr H)$ is isomorphic to $B \wr N_{\mathrm{Sym}(n)}(H)$. (You may use the result of Exercise 125(a).)

Exercise 127. Let m be a non-negative integer. Construct a finite group G with trivial centre for which there are exactly $m+1$ non-isomorphic groups in the automorphism tower $G = G_0 \to G_1 \to G_2 \to \ldots$ (with $G_{i+1} = \operatorname{Aut} G_i$ for each i). (You may use the result of Exercise 126(b).)

Exercise 128. Construct a group G with trivial centre for which all groups G_i $(i < \infty)$ in the automorphism tower are pairwise non-isomorphic. (This takes some work. It helps if you did the previous exercises.)

Exercise 129. Let n be a positive integer, let G be a group, and let $S_0, S_1, \ldots, S_n$ be subgroups of G such that, for each $i < n$, the group S_i is a normal subgroup of S_{i+1} with $C_{S_{i+1}}(S_i) = \{1\}$. Prove: $C_{S_n}(S_0) = \{1\}$.

Exercise 130. Let n be a positive integer, and let G be the semi-direct product $(\mathbf{Z}/n\mathbf{Z}) \rtimes (\mathbf{Z}/n\mathbf{Z})^*$, where $(\mathbf{Z}/n\mathbf{Z})^*$ acts on $\mathbf{Z}/n\mathbf{Z}$ by multiplication.

(a) Prove: n is odd $\Leftrightarrow \#Z(G) = 1 \Leftrightarrow \#[G,G] = n \Rightarrow H^1((\mathbf{Z}/n\mathbf{Z})^*, \mathbf{Z}/n\mathbf{Z}) = 0$.

(b) Prove: n is odd $\Leftrightarrow G$ is complete (as defined in Exercise 106).

Exercise 131. (a) Prove that there is, up to isomorphism, exactly one infinite group that is generated by two elements of order 2.

(b) In class we defined, for each group G and each ordinal number α, a group G_α by $G_0 = G$, $G_{\alpha+1} = \operatorname{Aut} G_\alpha$, $G_\lambda = \lim_{\alpha<\lambda} G_\alpha$ (for a limit ordinal λ). Compute all groups G_α when G is the group from (a).

Exercise 132. Let $\Phi(G)$ denote the Frattini subgroup of a group G.

(a) Let G be a finite group. Prove: if $G/\Phi(G)$ is nilpotent, then G is nilpotent.

(b) Give an example of a group G that is not nilpotent but for which $G/\Phi(G)$ is nilpotent.