Topics in group theory: exercises
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Exercise 37. The *derived length* of a solvable group G is the least non-negative integer n for which the subgroup $G^{(n)}$ of G defined in Exercise 22 equals $\{1\}$. The *nilpotency class* of a nilpotent group G is the least non-negative integer n for which the subgroup $Z_n(G)$ of G defined in Exercise 24 equals G.

(a) Prove that each nilpotent group G is solvable, and that its derived length is at most its nilpotency class.

(b) Prove that for each non-negative integer n there is a finite group of nilpotency class n.

(c) Construct a group G with the property that $Z_n(G) \neq Z_m(G)$ for all non-negative integers n, m with $n \neq m$.

Exercise 38. A subgroup H of a group G is called *subnormal* if there exist a non-negative integer t and a chain $H_0 \subset H_1 \subset \ldots \subset H_t$ of subgroups of G such that $H = H_0$, $G = H_t$, and such that H_i is a normal subgroup of H_{i+1} for $0 \leq i < t$.

Suppose that G is a finite group, and that S is a subnormal Sylow subgroup of G. Prove that S is a normal subgroup of G.

Exercise 39. Let G be a group. In class we defined an element $\sigma \in G$ to be a *non-generator* of G if for every subset S of G that generates G, the set $S \backslash \{\sigma\}$ also generates G. Using Zorn's lemma, we proved that the set $\Phi(G)$ of non-generators of G is a subgroup of G. Prove this directly, not using Zorn's lemma.

Notation. In the following exercises, $\Phi(G)$ is as defined in Exercise 39. It is called the *Frattini subgroup* of G.

Exercise 40. Let G be a group. A G-set X is called *primitive* if it is transitive with $\#X > 1$ and the only block B of X with $\#B > 1$ is $B = X$.

(a) Let $\sigma \in G$. Prove: σ belongs to $\Phi(G)$ if and only if σ acts as the identity on each primitive G-set.

(b) Prove that one has $\Phi(G) \neq G$ if and only if there exists a primitive G-set.

Exercise 41. Let G be an *abelian* group.

(a) Prove: a subgroup $H \subset G$ is maximal if and only if $(G : H)$ is a prime number.

(b) Suppose G is multiplicatively written. Prove: $\Phi(G) = \bigcap_p G^p$, with p ranging over the set of prime numbers and $G^p = \{\sigma^p : \sigma \in G\}$.

Exercise 42. A group G is called *divisible* if for each positive integer n the map $G \to G$, $\sigma \mapsto \sigma^n$, is surjective.

(a) Prove that an abelian group G satisfies $\Phi(G) = G$ if and only if G is divisible.

(b) Determine the Frattini subgroup of each of the following groups: the additive groups of $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$, and the multiplicative groups $\mathbf{Q}^*$, $\mathbf{R}^*$, and $\mathbf{C}^*$.

Exercise 43. (a) Let k be a field, let V be a k-vector space of dimension greater than 1, and let $\operatorname{Aut} V$ be the set of k-linear automorphisms of V. Prove that the set of one-dimensional subspaces of V is a primitive $(\operatorname{Aut} V)$-set, with the natural action.

(b) Show that there is a divisible group G with $\Phi(G) \neq G$.

Exercise 44. (a) Let G be a group with center $Z(G)$, and let H and I be subgroups of G with $H \cap Z(G) \subset I \subset Z(G)$. Prove that there is a subgroup J of G with $H \subset J$ and $J \cap Z(G) = I$.

(b) Let G be the subgroup of $\mathrm{Gl}(3, \mathbf{Q})$ consisting of all upper triangular matrices with 1's on the diagonal. What is the center of G? Is G nilpotent? Is G divisible? What is $\Phi(G)$?

Exercise 45. Construct a non-divisible group G with $\Phi(G) = G$. (*Warning*: this is hard.)

Exercise 46. Let G_0, G_1 be groups, and let $f\colon G_0 \to G_1$ be a surjective group homomorphism.

(a) Prove: $f(\Phi(G_0)) \subset \Phi(G_1)$, and if G_0 is a finite nilpotent group, then $f(\Phi(G_0)) = \Phi(G_1)$.

(b) Give an example in which G_0 is abelian and $f(\Phi(G_0)) \neq \Phi(G_1)$.

(c) Give an example in which G_0 is a finite solvable group and $f(\Phi(G_0)) \neq \Phi(G_1)$.

Exercise 47. Let G be a finite group. Prove: G is nilpotent if and only if each subgroup of G is subnormal (as defined in Exercise 38).

Exercise 48. In class we proved: if G is a finite group with Frattini subgroup $\Phi(G)$, and d is the smallest cardinality of a set of generators of G, then the order of the kernel of the natural map $\operatorname{Aut} G \to \operatorname{Aut}(G/\Phi(G))$ divides $(\#\Phi(G))^d$. Prove that for any two positive integers n, m, equality holds when G is taken to be the direct sum of n copies of $\mathbf{Z}/m\mathbf{Z}$.