Topics in group theory: exercises

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Exercise 85. Let p be a prime number and let G be a non-abelian p-group. Show that $0 \to Z(G) \to G \to G/Z(G) \to 0$ is an exact sequence which is not semisplit.

Exercise 86. Let G be a group with two normal subgroups K and N, and let H/N be a complement of the normal subgroup KN/N of G/N. Suppose also that $H \cap K$ has a complement in H. Show that K has a complement in G.

Exercise 87. Let G be a finite group and let N be a normal subgroup. Suppose that $\mathrm{Inn}(N)$ has a complement in $\mathrm{Aut}(N)$, and suppose that $\#Z(N)$ is coprime to $[G : N]$. Show that N has a complement in G.

Exercise 88. Let G be a finite group and let N be a normal subgroup such that $\#N$ is coprime to $[G : N]$. Show that N is a characteristic subgroup of G.

Exercise 89. Show that for every field K the exact sequence

$$0 \to \mathrm{SL}_2(K) \to \mathrm{GL}_2(K) \to K^* \to 0$$

is semisplit. Is the sequence

$$0 \to \{1, -1\} \to \mathrm{SL}_2(K) \to \mathrm{PSL}_2(K) \to 0$$

semisplit for all fields K of characteristic not 2?

Exercise 90. Let G be a finite group, N a normal subgroup and let p be a prime number which does not divide $[G : N]$. Show that there is a subgroup H of G such that p does not divide $\#H$ and $HN = G$.

Exercise 91. Give an example of a group G and a normal subgroup N such that N has a complement in G, but this complement is not unique up to conjugation in G.

Exercise 92. Let G be a group. Show that the semidirect product $G \rtimes G$, where G acts on G by conjugation, is isomorphic to $G \times G$.

Exercise 93. Show that up to isomorphism there are exactly 4 groups of order 30.

Exercise 94. We say that a finite group G is p-nilpotent for a prime number p if it has a normal subgroup N such that $\#N$ is coprime to $[G : N]$ and $[G : N]$ is a power of p. Show that a finite group G is nilpotent if and only if it is p-nilpotent for every prime p.

Exercise 95. Let p be a prime number and let G be a finite nonabelian p-group that has an abelian subgroup of index p. Show that the number of abelian subgroups of index p in G is either 1 or $p + 1$.

Exercise 96. Show that for every group G we have $Z(G) \cap [G, G] \subset \Phi(G)$.