An algebraic proof of Brauer's class number relations

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Abstract. For any non-cyclic Galois extension of number fields the regulators and class numbers of the intermediate fields satisfy the so-called Brauer relations. These relations, which are given in terms of the combinatorics of the Galois group, can be easily proved with the analytic class number formula. By using cohomological class field theory, we give a purely algebraic proof, which gives information about the structure of the groups involved, and not just about their order or measure.

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1. Introduction

Let G be a finite group that acts on a number field K by field automorphisms. By a G-set we mean a finite set with a left-action of G. For a G-set X the set K_X of G-equivariant maps $X \to K$ has the structure of a finite étale $\mathbb{Q}$-algebra, i.e., a product of number fields. If F is a finite product of number fields, then we write $h(F)$ and $R(F)$ for the products of the class numbers and of the regulators of the components of F, and $w(F)$ for the order of the torsion subgroup $\mu(F)$ of F^*.

Throughout this paper X and Y will denote two G-sets so that every $g \in G$ has the same number of fix points on X and on Y. Such G-sets are said to be *linearly equivalent*. In this note we present a purely algebraic proof of the equality

$$\frac{h(K_X)R(K_X)}{w(K_X)} = \frac{h(K_Y)R(K_Y)}{w(K_Y)}, \tag{1.1}$$

which is commonly referred to as "Brauer's class number relations." The main ingredients of the proof are the description of the second Galois cohomology group of the idele class group due to Nakayama and Hochschild, and a result in integral representation theory known as Conlon's induction theorem.

A much easier proof of this result has been available for nearly 40 years already: Brauer [2] and Kuroda [13] showed independently how Artin's formalism of L-functions implies that K_X and K_Y have the same zeta-function. Using the formula for the residue at $s = 1$ of zeta-functions of number fields, it is not hard to derive (1.1).

The search for an algebraic proof, i.e., a proof not relying on the residue formula for the zeta function, has a long tradition as well, starting with Hilbert's algebraic proof [10] in 1894 of a special case of (1.1) shown analytically by Dirichlet in 1842

[7]. For the case that $G = V_4$ algebraic proofs have been published by Kubota [12] in 1953 when the base field is $\mathbb{Q}$ and by Lemmermeyer [15] in 1994 for arbitrary base fields. The present paper grew out of [5] where an algebraic proof is given for nilpotent groups G.

The algebraic approach used in this paper actually gives rise to more detailed results about the relation between the group structures of the class groups of K_X and K_Y, and not just the orders of these groups. See [8, 1] for precise results. The present paper is devoted only to giving an algebraic proof of (1.1).

2. Outline of the proof

For a commutative ring R with 1 we denote the group ring of G over R by RG or $R[G]$, and for any G-set T we define the left RG-module $RT = R[T]$ as the free R-module generated by T with G permuting T. The fact that X and Y are linearly equivalent implies that $\mathbb{Q}X$ is isomorphic to $\mathbb{Q}Y$ as a $\mathbb{Q}G$-module; see [14, Chap. XVIII, Th. 2.3]. This means that we can choose an injective $\mathbb{Z}G$-linear homomorphism

$$\varphi\colon \mathbb{Z}X \longrightarrow \mathbb{Z}Y$$

with a finite cokernel.

Suppose that $M \to N$ is a $\mathbb{Z}G$-module homomorphism. We define $\mathrm{Cok}_M^N(X)$ to be the cokernel of the induced homomorphism $\mathrm{Hom}_G(\mathbb{Z}X, M) \to \mathrm{Hom}_G(\mathbb{Z}X, N)$. The homomorphism φ induces a group homomorphism $\mathrm{Cok}_M^N(\varphi)\colon\ \mathrm{Cok}_M^N(Y) \to \mathrm{Cok}_M^N(X)$. Thus, we have the following diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_G(\mathbb{Z}Y, M) & \to & \mathrm{Hom}_G(\mathbb{Z}Y, N) & \to & \mathrm{Cok}_M^N(Y) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Cok}_M^N(\varphi)} & & \\
\mathrm{Hom}_G(\mathbb{Z}X, M) & \to & \mathrm{Hom}_G(\mathbb{Z}X, N) & \to & \mathrm{Cok}_M^N(X) & \to & 0.
\end{array}$$

If $\mathrm{Cok}_M^N(\varphi)$ has finite kernel and cokernel, then we write

$$\chi_M^N(\varphi) = \frac{\#\,\mathrm{Cok}\ \mathrm{Cok}_M^N(\varphi)}{\#\,\mathrm{Ker}\ \mathrm{Cok}_M^N(\varphi)}.$$

In may instances where we use this notation, the map $M \to N$ in inclusion map. For example, $\mu(K_X) = \mathrm{Cok}_0^{\mu(K)}(\mathbb{Z}X)$, so $\chi_0^{\mu(K)}(\varphi) = w(K_X)/w(K_Y)$.

To obtain a map between class groups is a bit more subtle: one can write the class group of K as the cokernel of the canonical map $J \to C$ where $J = J(K)$ is the group of ideles of K whose coordinates at finite primes are local units, and $C = C(K)$ is the idele class group of K. Since for any subgroup H of G the canonical maps $J(K^H) \to$

$J(K)^H$ and $C(K^H) \to C(K)^H$ are isomorphisms, we have $\mathrm{Cok}_J^C(\mathbb{Z}X) = \mathrm{Cl}(K_X)$, and φ induces a homomorphism $\mathrm{Cok}_J^C(\varphi)$: $\mathrm{Cl}(K_Y) \to \mathrm{Cl}(K_X)$. This map between class groups is not the map we will use in the proof because we do not want to separate the class group and the units.

We write U for the group of finite unit ideles, i.e., the subgroup of J consisting of those ideles whose coordinates at finite primes are units, and whose coordinates at infinite primes are 1. This group U is the unit group of the profinite completion of the ring of integers of K. We will call the cokernel of the map $U \to C$ the *padded Picard group* $P(K)$ of K. Writing S for the maximal compact subgroup of $(K \otimes \mathbb{R})^*$, the usual Picard group of K is the cokernel of $U \times S \to C$, and it is the standard algebraic object to consider when one wants to study the unit group and class group in an integrated fashion (see e.g. [16, Chap. III, §1]). We use the padded Picard group in this paper because it has slightly better cohomological properties.

We define padded Picard groups of finite étale algebras over $\mathbb{Q}$ by taking products over the components, and with this definition one see that there is a canonical isomorphism $P(K_X) \cong \mathrm{Cok}_U^C(\mathbb{Z}X)$. In particular, φ induces a homomorphism $\mathrm{Cok}_U^C(\varphi)$: $P(K_Y) \to P(K_X)$. We will show that $\mathrm{Cok}_U^C(\varphi)$ has finite kernel and cokernel. Thus, $\chi_U^C(\varphi)$ is a well-defined positive rational number, and the proof of (1.1) essentially consists of two different ways to compute this number.

We write $L(F) = h(F)R(F)/w(F)$ for any finite étale $\mathbb{Q}$-algebra F, and we let d be the degree of K^G over $\mathbb{Q}$. We first state three propositions which will be proved independently.

(2.1) Proposition. *Let E be the cokernel of φ. The homomorphism $\mathrm{Cok}_U^C(\varphi)$ has finite kernel and cokernel and we have*

$$\frac{L(K_X)}{L(K_Y)} = \frac{(\#E)^d}{\#(E^G)} \cdot \chi_U^C(\varphi).$$

This result implies that the left-hand side, which is a priori only a positive real number, is in fact a rational number. The proof of (2.1), which is mostly an exercise with regulators, is the subject of Section 3.

(2.2) Corollary. *If p is a prime number for which $\mathbb{Z}_pX$ and $\mathbb{Z}_pY$ are isomorphic as $\mathbb{Z}_pG$-modules, then $\mathrm{ord}_p(L(K_X)/L(K_Y)) = 0$.*

To see how this follows from (2.1) note that under the condition of (2.2) we can choose φ so that E has order coprime to p. Then φ and therefore $\mathrm{Cok}_U^C(\varphi)$ become isomorphisms when localized at p, so that $\mathrm{Cok}_U^C(\varphi)$ has kernel and cokernel of order coprime to p. Applying (2.1) with this choice of φ gives (2.2).

The next result will allow us to replace U by a finite index subgroup with much better cohomological properties. Recall that a $\mathbb{Z}[G]$-module M is said to be cohomologically trivial if the Tate cohomology group $\hat{H}^i(H, M)$ vanishes for all subgroups H of G and all $i \in \mathbb{Z}$; see [3, p. 111].

(2.3) Proposition. *There is a G-stable subgroup V of finite index in U so that*

(1) *V is cohomologically trivial;*
(2) $\chi_V^C(\varphi) = \chi_U^C(\varphi)$.

The proof is given in Section 4. It uses the conductor discriminant product formula in the form of a result of Fröhlich that says that as a $\mathbb{Z}[G]$-module the ring of integers of K is "factor equivalent" to a free module [9, 6].

From now on, let V be as in (2.3). For any finite $\mathbb{Z}G$-module M the homomorphism $V \to C$ induces homomorphisms

$$f_M^0 = \mathrm{Hom}_G(M, V) \to \mathrm{Hom}_G(M, C), \qquad f_M^1 = \mathrm{Ext}_G^1(M, V) \to \mathrm{Ext}_G^1(M, C).$$

We will show in Section 4 that these maps have finite kernel and cokernel and we define

$$\psi(M) = \frac{\#\,\mathrm{Cok}\, f_M^1}{\#\,\mathrm{Ker}\, f_M^1} \cdot \frac{\#\,\mathrm{Ker}\, f_M^0}{\#\,\mathrm{Cok}\, f_M^0}.$$

Thus, $\psi(M)$ is a positive rational number built up from primes dividing $\#M$. We will see in Section 4 that the fact that V is cohomologically trivial easily implies the next Proposition.

(2.4) Proposition. *For $E = \mathrm{Cok}\,\varphi$ the homomorphisms f_E^0 and f_E^1 have finite kernels and cokernels, and we have $\chi_V^C(\varphi) = \psi(E)$.*

Combining the three Propositions, it follows that

$$\frac{L(K_X)}{L(K_Y)} = \frac{(\#E)^d}{\#(E^G)} \cdot \psi(E). \tag{2.5}$$

In particular $L(K_X)/L(K_Y)$ only depends on K and the finite $\mathbb{Z}G$-module $E = \mathrm{Cok}\,\varphi$ obtained from φ, but not on X and Y themselves.

By considering a maximal filtration of E with sub-$\mathbb{Z}G$-modules and applying homological algebra and the description of the second Galois cohomology group of C, we will deduce the following special case of (1.1) in Section 5. For a prime number p we say that a finite group H is *cyclic modulo p* if it has a normal subgroup N so that the quotient H/N is cyclic and N is a p-group.

(2.6) Proposition. *If p is a prime number and G is cyclic modulo p, then we have*

$$\mathrm{ord}_p\left(\frac{L(K_X)}{L(K_Y)}\right) = 0.$$

It turns out that this Proposition, together with Conlon's theory of permutation representations suffice to obtain the general result (1.1). This is the subject of Section 6.

3. Regulator computations

The goal of this section is to prove (2.1). For a G-set T we define

$$j(T) = \prod_{O\in G\backslash T} \#G/\#O,$$

where $G\backslash T$ denotes the set of G-orbits of T.

(3.1) Lemma. *We have*

$$\begin{array}{llcl}
(1) & \chi_0^{\mathbb{Z}G}(\varphi) & = & \#E; \\
(2) & \chi_0^{\mathbb{Z}}(\varphi) & = & \#(E_G); \\
(3) & j(X)/j(Y) & = & \#(E_G)/\#(E^G).
\end{array}$$

Proof. We will use homology, cohomology and Tate-cohomology groups as in [3, p. 102]. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & (\mathbb{Z}X)_G & \to & (\mathbb{Z}Y)_G & \to & E_G & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle t} & & \\
0 & \to & (\mathbb{Z}X)^G & \to & (\mathbb{Z}Y)^G & \to & E^G & \to & 0,
\end{array}$$

where the vertical maps are given by the action of the trace element $t = \sum_{g\in G} g \in \mathbb{Z}G$. Since $H_1(G, E)$ is a torsion group, and $(\mathbb{Z}X)_G$ is torsion free, the top row is of the diagram is exact. The two left most vertical maps are injective with cokernels $\hat{H}^0(G, \mathbb{Z}X)$ and $\hat{H}^0(G, \mathbb{Z}Y)$. Since $H^1(H, \mathbb{Z}) = 0$ for any finite group H, Shapiro's lemma [3, p. 99] implies that $H^1(G, \mathbb{Z}X) = 0$, so the bottom row is also exact. Since $j(T) = \#\hat{H}^0(G, \mathbb{Z}T)$ for any G-set T, we see with the snake lemma that

$$\frac{\#(E_G)}{\#(E^G)} = \frac{\#\,\mathrm{Ker}\,t}{\#\,\mathrm{Cok}\,t} = \frac{\#\hat{H}^0(G, \mathbb{Z}X)}{\#\hat{H}^0(G, \mathbb{Z}Y)} = \frac{j(X)}{j(Y)}.$$

This shows (3). Note that $\mathrm{Cok}_0^{\mathbb{Z}G}(\varphi)$ is $\mathbb{Z}$-dual of φ, so that it is injective with cokernel of order $\#E$. It remains to show (2). The map $\mathrm{Cok}_0^{\mathbb{Z}}(\varphi)$ is the $\mathbb{Z}$-dual of the homomorphism $(\mathbb{Z}X)_G \to (\mathbb{Z}Y)_G$ of free abelian groups, which we just saw is injective with cokernel E_G. Thus, (2) follows. □

Let us recall the definition of the regulator. Let F be a finite étale $\mathbb{Q}$-algebra with ring of integers $\mathcal{O}_F$. By an infinite prime $\mathfrak{p}$ of F we mean an infinite prime $\mathfrak{p}$ of a component F_0. Such a prime $\mathfrak{p}$ defines a map $||\cdot||_{\mathfrak{p}}\colon F \to \mathbb{R}_{\geq 0}$ which is defined by first projecting to F_0 and then taking the normalized absolute value. Let $S(F)$ be a set of infinite primes of F which for every component F_0 of F contains all but exactly one infinite prime of F_0. The traditional definition of the regulator $R(\mathcal{U})$of a subgroup $\mathcal{U}$ of $\mathcal{O}_F^*$ of finite index is the following: take $\epsilon_1, \ldots, \epsilon_r \in \mathcal{U}$ that generate $\mathcal{U}/\mathcal{U}_{\mathrm{tors}}$, and let $R(\mathcal{U})$ be the absolute value of the determinant of the real $r \times r$-matrix $(\log ||\epsilon_i||_{\mathfrak{p}_j})_{i,j}$, where $S(F) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

Throughout this section will will write (M, N) for $\mathrm{Hom}_G(M, N)$ when M and N are $\mathbb{Z}G$-modules. For any G-set T and G-stable subgroup $\mathcal{U}$ of finite index in $\mathcal{O}_K^*$ we write

$$\tilde{R}(T,\mathcal{U}) = \frac{R((\mathbb{Z}T,\mathcal{U}))}{\#(\mathbb{Z}T,\mathcal{U})_{\mathrm{tors}}},$$

where the regulator is computed in the étale $\mathbb{Q}$-algebra $K_T = (\mathbb{Z}[T], K)$.

(3.2) Proposition. *Let $\mathcal{U}$ be a G-stable subgroup of finite index in $\mathcal{O}_K^*$. Then $\mathrm{Cok}_{\mathcal{U}}^{(K\otimes\mathbb{R})^*}(\varphi)$ has finite kernel and cokernel and we have*

$$\chi_{\mathcal{U}}^{(K\otimes\mathbb{R})^*}(\varphi) = \frac{\#(E^G)}{(\#E)^d} \cdot \frac{\tilde{R}(Y,\mathcal{U})}{\tilde{R}(X,\mathcal{U})}.$$

Proof. The first step of the proof is to show that the validity of (3.2) does not depend on the choice of $\mathcal{U}$. Suppose that $\mathcal{V} \subset \mathcal{U}$ is a sub-$\mathbb{Z}G$-module of finite index. For any subgroup H of G we have We have $R(\mathcal{V}^H)/R(\mathcal{U}^H) = [\mathcal{V}^H/\mathcal{V}^H_{\mathrm{tors}} : \mathcal{U}/\mathcal{U}^H_{\mathrm{tors}}]$, which by the snake lemma implies that

$$\frac{R(\mathcal{V}^H)}{\#\mathcal{V}^H_{\mathrm{tors}}} = [\mathcal{U}^H : \mathcal{V}^H]\frac{R(\mathcal{U}^H)}{\#\mathcal{U}^H_{\mathrm{tors}}}.$$

For any G-set T we deduce that

$$\tilde{R}(T, \mathcal{V}) = [(\mathbb{Z}T,\mathcal{U}) : (\mathbb{Z}T, \mathcal{V})]\ \tilde{R}(T,\mathcal{U}). \tag{3.3}$$

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & (\mathbb{Z}Y, \mathcal{V}) & \to & (\mathbb{Z}X, \mathcal{V}) & \to & \bullet & \to & 0 \\
 & & \cap & & \cap & & \big\downarrow \mathrm{Cok}_{\mathcal{V}}^{\mathcal{U}}(\varphi) & & \\
0 & \to & (\mathbb{Z}Y,\mathcal{U}) & \to & (\mathbb{Z}X,\mathcal{U}) & \to & \bullet & \to & 0.
\end{array}$$

By first using (3.3) for $T = X$ and for $T = Y$, and then the snake lemma for the diagram above, we see that

$$\frac{\tilde{R}(Y,\mathcal{V})}{\tilde{R}(X,\mathcal{V})}\,\chi_{\mathcal{V}}^{(K\otimes\mathbb{R})^*}(\varphi) = \frac{\tilde{R}(Y,\mathcal{U})}{\tilde{R}(X,\mathcal{U})}\,\frac{[(\mathbb{Z}Y,\mathcal{U}):(\mathbb{Z}Y,\mathcal{V})]}{[(\mathbb{Z}X,\mathcal{U}):(\mathbb{Z}X,\mathcal{V})]}\,\chi_{\mathcal{V}}^{(K\otimes\mathbb{R})^*}(\varphi)$$
$$= \frac{\tilde{R}(Y,\mathcal{U})}{\tilde{R}(X,\mathcal{U})}\,\frac{\chi_{\mathcal{V}}^{(K\otimes\mathbb{R})^*}(\varphi)}{\chi_{\mathcal{V}}^{\mathcal{U}}(\varphi)} = \frac{\tilde{R}(Y,\mathcal{U})}{\tilde{R}(X,\mathcal{U})}\,\chi_{\mathcal{U}}^{(K\otimes\mathbb{R})^*}(\varphi).$$

This shows that (3.2) holds for all $\mathcal{U}$ if and only if it holds for $\mathcal{U} = \mathcal{O}_K^*$.

It remains to show (3.2) for a single choice of $\mathcal{U}$, and we will take $\mathcal{U} = (\mathcal{O}_K^*)^{2w(K)}$. Then $\mathcal{U}$ is G-stable and we have

(1) $\mathcal{U}$ has no torsion;

(2) for every field embedding $\sigma\colon K \to \mathbb{C}$ we have $\sigma(\mathcal{U}) \cap \mathbb{R} \subset \mathbb{R}_{>0}$.

To see that (2) holds, note that squaring gives and isomorphism $(\mathcal{O}_K^*)^{w(K)} \xrightarrow{\sim} \mathcal{U}$ and take invariants under complex conjugation.

The exponential map $K \otimes \mathbb{R} \to (K \otimes \mathbb{R})^*$ is given by $x \mapsto \sum_{n=0}^{\infty} x^n/n!$. This map is a homomorphism of groups, whose image is the set of elements of $(K \otimes \mathbb{R})^*$ which are positive on all real components of $K \otimes \mathbb{R}$. In particular, its cokernel has order 2^s, where s is the number of real primes of K.

Define $\log \mathcal{U}$ to be the inverse image in $K \otimes \mathbb{R}$ of the subgroup $\mathcal{U}$ of $(K \otimes \mathbb{R})^*$. The fact that $\mathcal{U}$ is torsion free and totally positive in the sense of (2) above, implies that the map $(\log \mathcal{U})^H \to \mathcal{U}^H$ is surjective for every subgroup H of G. One deduces that for every G-set T the group $\mathrm{Cok}_{\log \mathcal{U}}^{K\otimes\mathbb{R}}(T)$ embeds as a subgroup of $\mathrm{Cok}_{\mathcal{U}}^{(K\otimes\mathbb{R})^*}(T)$ of index $2^{s(T)}$ where $s(T)$ is the number of real primes of K_T. Since $s(X) = s(Y)$ it follows that

$$\chi_{\mathcal{U}}^{(K\otimes\mathbb{R})^*}(\varphi) = \chi_{\log \mathcal{U}}^{K\otimes\mathbb{R}}(\varphi). \tag{3.4}$$

Viewing $\mathbb{Z}$ as a subring of $K \otimes \mathbb{R}$ we define a lattice

$$W = \mathbb{Z} \oplus \log \mathcal{U} \subset K \otimes \mathbb{R},$$

which is discrete and cocompact in $K \otimes \mathbb{R}$. We will compare the lattice W to a standard lattice $\Lambda \subset K \otimes \mathbb{R}$ which is defined as follows. For an infinite prime $\mathfrak{p}$ of K let $\Lambda_{\mathfrak{p}} = \mathbb{Z} \subset \mathbb{R} = K_{\mathfrak{p}}$ when $\mathfrak{p}$ is real, and let $\Lambda_{\mathfrak{p}} = \mathbb{Z} \oplus \frac{1+2\pi i}{2}\mathbb{Z} \subset \mathbb{C} \cong K_{\mathfrak{p}}$ when $\mathfrak{p}$ is complex (this does not depend on the choice of the topological field isomorphism $\mathbb{C} \cong K_{\mathfrak{p}}$). Setting $\Lambda = \prod_{\mathfrak{p}} \Lambda_{\mathfrak{p}} \subset \prod_{\mathfrak{p}} K_{\mathfrak{p}} = K \otimes \mathbb{R}$ we obtain a G-stable discrete cocompact lattice in $K \otimes \mathbb{R}$. It is easy to check that Λ is free as a $\mathbb{Z}[G]$-module of rank $d = [K^G : \mathbb{Q}]$.

For any Haar measure on $K \otimes \mathbb{R}$ we now claim that

$$\frac{\operatorname{covol} W}{\operatorname{covol} \Lambda} = [K:\mathbb{Q}]R(\mathcal{U}).$$

To see this, consider the kernel of the exponential map

$$I = \bigoplus_{\mathfrak{p} \text{ complex}} 2\pi i\mathbb{Z} \subset \bigoplus_{\mathfrak{p}|\infty} K_{\mathfrak{p}} = K \otimes \mathbb{R},$$

which is contained in both W and in Λ. Clearly, Λ/I is torsion free, and since $\mathcal{U}$ has no torsion, W/I is also torsion free. In order to compare the covolumes of W and Λ we can therefore do an orthogonal projection onto the subspace $\prod_{\mathfrak{p}} \mathbb{R} \subset \prod_{\mathfrak{p}} K_{\mathfrak{p}} = K \otimes \mathbb{R}$. Suppose that K has r_1 real primes and r_2 complex primes, and choose some numbering $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of them with the real primes first and $r = r_1 + r_2$. Then the projection of the lattice Λ is $\mathbb{Z}^{r_1} \oplus (\frac{1}{2}\mathbb{Z})^{r_2}$. For an infinite prime $\mathfrak{p}$ of K let $|\cdot|_{\mathfrak{p}}$ be the absolute value of $K_{\mathfrak{p}}$ extending the usual abolute value on $\mathbb{R}$, and let $||\cdot||_{\mathfrak{p}} = |\cdot|_{\mathfrak{p}}^{[K_{\mathfrak{p}}:\mathbb{R}]}$. The image of W is generated by the vector $(1, \ldots, 1)$, and the vectors $(\log|\epsilon_i|_{\mathfrak{p}_j})_{j=1}^r$ where $\epsilon_1, \ldots, \epsilon_{r-1}$ is a $\mathbb{Z}$-basis for $\mathcal{U}$. By some row operations and the product formula $\sum_{\mathfrak{p}|\infty} \log||\epsilon||_{\mathfrak{p}} = 0$ for $\epsilon \in \mathcal{O}_K^*$, we deduce that

$$\begin{aligned}
\frac{\operatorname{covol} W}{\operatorname{covol} \Lambda} &= 2^{r_2} \begin{vmatrix} 1 & \log|\epsilon_1|_{\mathfrak{p}_1} & \cdots & \log|\epsilon_{r-1}|_{\mathfrak{p}_1} \\ \vdots & \vdots & & \vdots \\ 1 & \log|\epsilon_1|_{\mathfrak{p}_r} & \cdots & \log|\epsilon_{r-1}|_{\mathfrak{p}_r} \end{vmatrix} \\
&= \begin{vmatrix} [K_{\mathfrak{p}_1}:\mathbb{R}] & \log||\epsilon_1||_{\mathfrak{p}_1} & \cdots & \log||\epsilon_{r-1}||_{\mathfrak{p}_1} \\ \vdots & \vdots & & \vdots \\ [K_{\mathfrak{p}_r}:\mathbb{R}] & \log||\epsilon_1||_{\mathfrak{p}_r} & \cdots & \log||\epsilon_{r-1}||_{\mathfrak{p}_r} \end{vmatrix} \\
&= (r_1 + 2r_2) \begin{vmatrix} \log||\epsilon_1||_{\mathfrak{p}_1} & \cdots & \log||\epsilon_{r-1}||_{\mathfrak{p}_1} \\ \vdots & & \vdots \\ \log||\epsilon_1||_{\mathfrak{p}_{r-1}} & \cdots & \log||\epsilon_{r-1}||_{\mathfrak{p}_{r-1}} \end{vmatrix} = [K:\mathbb{Q}]\ R(\mathcal{U}).
\end{aligned}$$

The next step is to obtain a version of this result for the étale algebras K_X and K_Y. If H is a subgroup of G then the standard isomorphism $(K \otimes \mathbb{R})^H = K^H \otimes \mathbb{R}$ identifies the standard lattice Λ^H with the standard lattice for K^H in $K^H \otimes \mathbb{R}$. Moreover, it identifies $W^H = (\mathbb{Z} \oplus (\log \mathcal{U})^H)$ with $\mathbb{Z} \oplus \log(\mathcal{U}^H)$, where this second "log" is the one for the field K^H. The equality above, applied now to K^H rather than K therefore gives

$$\frac{\operatorname{covol}(W^H)}{\operatorname{covol}(\Lambda^H)} = [K^H:\mathbb{Q}]\ R(\mathcal{U}^H) = \frac{[K:\mathbb{Q}]}{j(G/H)} R(\mathcal{U}^H).$$

It follows that for any G-set T we have

$$\frac{\operatorname{covol}(\mathbb{Z}T, W)}{\operatorname{covol}(\mathbb{Z}T, \Lambda)} = \frac{R(\mathbb{Z}T, \mathcal{U}) \cdot [K:\mathbb{Q}]^{\#(G\backslash T)}}{j(T)}$$

for any Haar measure on the real vector space $(\mathbb{Z}T, K \otimes \mathbb{R})$, where we define $j'(T)$ to be $[K:\mathbb{Q}]^{\#(G\backslash T)}/j(T)$. Now consider the diagram

$$\begin{array}{ccccc}
(\mathbb{Z}Y, W) & \subset & (\mathbb{Z}Y, K\otimes\mathbb{R}) & \supset & (\mathbb{Z}Y, \Lambda) \\
\big\downarrow {\scriptstyle \operatorname{Cok}_0^W(\varphi)} & & \big\downarrow {\scriptstyle \cong} & & \big\downarrow {\scriptstyle \operatorname{Cok}_0^\Lambda(\varphi)} \\
(\mathbb{Z}X, W) & \subset & (\mathbb{Z}X, K\otimes\mathbb{R}) & \supset & (\mathbb{Z}X, \Lambda).
\end{array}$$

Choose a Haar measure on $(\mathbb{Z}X, K \otimes \mathbb{R})$ and consider the four lattices as lattices in this real vector space. Then we have

$$\begin{aligned}
\chi_0^W(\varphi) &= \frac{\operatorname{covol}(\mathbb{Z}Y, W)}{\operatorname{covol}(\mathbb{Z}X, W)} = \frac{j(X)[K:\mathbb{Q}]^{\#(G\backslash Y)} R(\mathbb{Z}Y,\mathcal{U})}{j(Y)[K:\mathbb{Q}]^{\#(G\backslash X)} R(\mathbb{Z}X,\mathcal{U})} \frac{\operatorname{covol}(\mathbb{Z}Y, \Lambda)}{\operatorname{covol}(\mathbb{Z}X, \Lambda)} \\
&= \frac{j(X)R(\mathbb{Z}Y,\mathcal{U})}{j(Y)R(\mathbb{Z}X,\mathcal{U})} \chi_0^\Lambda(\varphi).
\end{aligned}$$

For the last equality we used that X and Y have the same number of G-orbits. Since Λ is a free $\mathbb{Z}G$-module of rank d, and $W = \mathbb{Z} \oplus \log \mathcal{U}$, we can deduce with Lemma (3.1) that

$$\chi_0^{\log\mathcal{U}}(\varphi) = \frac{\chi_0^W(\varphi)}{\chi_0^{\mathbb{Z}}(\varphi)} = \frac{(\chi_0^{\mathbb{Z}G}(\varphi))^d}{\chi_0^{\mathbb{Z}}(\varphi)} \frac{j(X)}{j(Y)} \frac{R(\mathbb{Z}Y,\mathcal{U})}{R(\mathbb{Z}X,\mathcal{U})} = \frac{\#E^d}{\#E^G} \frac{R(\mathbb{Z}Y,\mathcal{U})}{R(\mathbb{Z}X,\mathcal{U})}.$$

We have $\chi_{\log\mathcal{U}}^{K\otimes\mathbb{R}}(\varphi)\chi_0^{\log\mathcal{U}}(\varphi) = \chi_0^{K\otimes\mathbb{R}}(\varphi) = 1$. With (3.4) this completes the proof of (3.2). ☐

Proof of (2.1). For every G-set T we have a short exact sequence

$$0 \to \operatorname{Cok}_{\mathcal{O}_K^*}^{(K\otimes\mathbb{R})^*}(T) \to \operatorname{Cok}_U^C(T) \to \operatorname{Cok}_J^C(T) \to 0.$$

Using (3.2) with $\mathcal{U} = \mathcal{O}_K^*$ and the fact that $\operatorname{Cok}_J^C(T)$ is the class group of K_T, we see that $\operatorname{Cok}_U^C(\varphi)$ has finite kernel and cokernel and that

$$\chi_U^C(\varphi) = \chi_{\mathcal{O}_K^*}^{(K\otimes\mathbb{R})^*}(\varphi)\chi_J^C(\varphi) = \frac{\#(E^G)}{(\#E)^d} \frac{L(K_X)}{L(K_Y)}.$$

This shows (2.1).

4. A cohomological expression

The goal of this section is to prove (2.3) and (2.4). Let p be a prime number, let $\mathcal{O}_p = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p = \prod_{\mathfrak{p}|p} \mathcal{O}_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the primes of K lying over p. Let $n = n_p \geq 0$ be an integer which is big enough to ensure that the exponential map defines an isomorphism

$$\exp_p\colon\ p^n\mathcal{O}_p \xrightarrow{\sim} 1 + p^n\mathcal{O}_p \qquad x \mapsto \sum_{k\geq 0} x^k/k!.$$

Since $\mathcal{O}_p \otimes \mathbb{Q}$ is a free module of finite rank over $\mathbb{Q}_pG$ the $\mathbb{Z}_pG$-module $\mathcal{O}_p$ has a free sub-$\mathbb{Z}_pG$-module $\Lambda_{\mathfrak{p}}$ of finite index.

(4.1) Lemma. *We have* $\chi^{\mathcal{O}_p^*}_{1+p^n\mathcal{O}_p}(\varphi) = 1$ *and* $\chi^{\mathcal{O}_p}_{\Lambda_p}(\varphi) = 1$.

Proof. The index of $1 + p^n\mathcal{O}_p(F)$ in $\mathcal{O}_p(F)^*$ is $p^{(n-1)[F:\mathbb{Q}]}$ times the product over all primes $\mathfrak{p}$ of F lying of p of $(1 - \frac{1}{N\mathfrak{p}})$. But for every integer q the algebras K_X and K_Y have the same number of primes $\mathfrak{q}$ with $N\mathfrak{q} = q$. See [6, p. 391] for a proof. Thus, $\mathrm{Cok}^{\mathcal{O}_p^*}_{1+p^n\mathcal{O}_p}(\varphi)$ is a map between finite groups of the same cardinality.

The second statement is a reformulation of Fröhlich's result that as a $\mathbb{Z}G$-module the ring of integers $\mathcal{O}_K$ is "factor equivalent" to a free module; see [9, 6]. More specifically, the map $\mathrm{Cok}^{\mathcal{O}_p}_{\Lambda_p}(\varphi)$ is a map between finite groups of the same cardinality by [6, Thm. 3.1]. $\square$

Proof of (2.3). We define V to be the product over all prime numbers p of groups V_p, where we take $V_p = \mathcal{O}_p^*$ if p is unramified in K, and $V_p = \exp_p(p^{n_p}\Lambda_p)$ for finitely many p that are ramified in K. Then V is a sub-$\mathbb{Z}G$-module of U of finite index. By using the fact that taking cohomology commutes with taking products, Shapiro's lemma, and the fact that in the local unramified case the group of local units is cohomologically trivial [3, p. 131], one sees that V is cohomologically trivial. By Lemma (4.1) we have $\chi^U_V(\varphi) = 1$ which gives $\chi^C_V(\varphi) = \chi^C_U(\varphi)$. This proves (2.3). $\square$

(4.2) Lemma *Let M be a finite $\mathbb{Z}[G]$-module and let i be a non-negative integer. Then the homomorphism*

$$f^i_M\colon\ \mathrm{Ext}^i_G(M, V) \to \mathrm{Ext}^i_G(M, C)$$

has finite kernel and cokernel.

Proof. Let n be a positive integer annihilating M. Recall that C/U is the padded Picard group P, which fits in an exact sequence

$$0 \to (K \otimes \mathbb{R})^*/\mathcal{O}_K^* \to P \to \mathrm{Cl}(K) \to 0.$$

Thus, P has finite n-torsion, and P/nP is also finite and we have short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & F_1 & \to & P & \to & nP & \to & 0 \\ 0 & \to & nP & \to & P & \to & F_2 & \to & 0, \end{array}$$

where the composite map $P \to nP \to P$ is multiplication by p, and F_1 and F_2 are finite $\mathbb{Z}[G]$-modules. Note that $\mathrm{Ext}^i_G(M, F)$ is finite if F is finite: if $A_\bullet$ is a resolution of M by finitely generated projective $\mathbb{Z}G$-module, then $\mathrm{Ext}^i_G(M, F)$ is the ith cohomology group of the complex $\mathrm{Hom}_G(A_\bullet, F)$, which consists of finite groups. If we apply the functors $\mathrm{Ext}^i_G(M, -)$ to the two short exact sequences, then we obtain two long exact sequences and we see that the maps $\mathrm{Ext}^i_G(M, P) \to \mathrm{Ext}^i_G(M, nP)$ and $\mathrm{Ext}^i_G(M, nP) \to \mathrm{Ext}^i_G(M, P)$ both have a finite kernel and a finite cokernel. Therefore, the composition $\mathrm{Ext}^i_G(M, P) \to \mathrm{Ext}^i_G(M, P)$ has a finite kernel and a finite cokernel. But this is the multiplication-by-n-map, and since $nM = 0$ it is the zero map. Thus, $\mathrm{Ext}^i_G(M, P)$ is finite. By writing down the long exact sequence for $0 \to U/V \to C/V \to C/U \to 0$, and using that $\mathrm{Ext}^i_G(M, U/V)$ is finite, we see that $\mathrm{Ext}^i_G(M, C/V)$ is finite for every i. With the long exact sequence associated to $0 \to V \to C \to C/V \to 0$ the lemma now follows. $\square$

Proof of (2.4). Since V is cohomologically trivial, we have $\mathrm{Ext}^1_G(\mathbb{Z}X, V) = 0$. It is a general result in idele cohomology that idele class groups have trivial first Galois cohomology, so we also have $\mathrm{Ext}^1_G(\mathbb{Z}X, C) = 0$. We now have the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathrm{Hom}_G(E, V) \arrow[r, "f_E^0"] \arrow[d] & \mathrm{Hom}_G(E, C) \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}_G(\mathbb{Z}Y, V) \arrow[r] \arrow[d] & \mathrm{Hom}_G(\mathbb{Z}Y, C) \arrow[r] \arrow[d] & \chi_V^C(Y) \arrow[r] \arrow[d, "\mathrm{Cok}_V^C(\varphi)"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_G(\mathbb{Z}X, V) \arrow[r] \arrow[d] & \mathrm{Hom}_G(\mathbb{Z}X, C) \arrow[r] \arrow[d] & \chi_V^C(X) \arrow[r] & 0. \\
 & \mathrm{Ext}^1_G(E, V) \arrow[r, "f_E^1"] \arrow[d] & \mathrm{Ext}^1_G(E, C) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

We just saw that f_E^0 and f_E^1 have finite kernels and cokernels. The snake lemma implies that $\chi_V^C(\varphi)$ has a finite kernel and cokernel as well, and we see that

$$\psi(E) = \frac{\#\,\mathrm{Cok}\, f_E^1}{\#\,\mathrm{Ker}\, f_E^1} \, \frac{\#\,\mathrm{Ker}\, f_E^0}{\#\,\mathrm{Cok}\, f_E^0} = \chi_V^C(\varphi).$$

Combining this result with (2.3) one sees that (2.4) holds. $\square$

5. Homological algebra and class field theory

The purpose of this section is to prove (2.6). Throughout this section, p denotes a prime number, and we assume that G is cyclic modulo p.

(5.1) Proposition. *Let G be a group that is cyclic modulo p and of order divisible by p, and let E be a simple $\mathbb{Z}_p G$-module. Then we have*

$$\operatorname{Ext}^2_G(E, V) = 0 \qquad \textit{and} \qquad \operatorname{Ext}^2_G(E, C) \cong E^G.$$

Proof. First consider the special case that $E = \mathbb{F}_p$, the cyclic group of order p with trivial G-action. By computing the long exact sequence of Ext-groups associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow E \longrightarrow 0,$$

we find a short exact sequence

$$0 \longrightarrow H^1(G, M) \otimes \mathbb{F}_p \longrightarrow \operatorname{Ext}^2_G(\mathbb{F}_p, M) \longrightarrow H^2(G, M)[p] \longrightarrow 0$$

for every $\mathbb{Z}G$-module M. Here the notation $[p]$ means that we take the p-torsion subgroup. Since V is cohomologically trivial, we see that $\operatorname{Ext}^2_G(\mathbb{F}_p, V) = 0$. A result of Nakayama and Hochschild, which is a corner stone of cohomological class field theory, tells us that $H^2(G, C)$ is cyclic of order $\#G$. See [11] or [3, p. 196/197] for details. Since G is assumed not to be cyclic, p divides the order of G, and we see that $H^2(G, C)[p]$ has order p. We have $H^1(G, C) = 0$, so the special case of (5.1) where $E = \mathbb{F}_p$ is proven.

Let N be a p-Sylow subgroup of G. Then N is normal in G and G/N is cyclic of order coprime to p. A simple $\mathbb{F}_p[G/N]$-module is simple as a $\mathbb{Z}_p[G]$-module, and in fact all simple $\mathbb{Z}_p[G]$-modules arise this way. To see this, note first that by Nakayama's lemma such simple modules are annihilated by p, and then use [4, Prop. (17.16)]. The ring $\mathbb{F}_p[G/N]$ is a product of finite fields $E_1 \times E_2 \times \cdots \times E_s$. Thus, $\mathbb{F}_p[G/N]$ is the direct sum of the system of representatives $\{E_1, E_2, \ldots, E_s\}$ of the isomorphism classes of all simple $\mathbb{Z}_p[G]$-modules. For every $\mathbb{Z}[G]$-module M we therefore have

$$\bigoplus_{i=1}^{s} \operatorname{Ext}^2_G(E_i, M) = \operatorname{Ext}^2_G(\mathbb{F}_p[G/N], M) = \operatorname{Ext}^2_N(\mathbb{F}_p, M).$$

By combining the special case we did already for the groups G and N, we see that $\operatorname{Ext}^2_G(E_i, V) = 0$ for all i and that $\operatorname{Ext}^2_G(E_i, C) = 0$, when E_i does not have trivial G-action. This completes the proof of (5.1). $\square$

We now proceed to compute $\psi(E)$ for simple $\mathbb{Z}_pG$-modules E.

(5.2) Lemma. *Suppose that G is cyclic module p, and that E is a finite $\mathbb{Z}_pG$-module which is either simple, or has trivial G-action. Then*

(1) *there exists a G-set T and a short exact sequence of $\mathbb{Z}_pG$-modules*

$$0 \longrightarrow \mathbb{Z}_pT \longrightarrow \mathbb{Z}_pT \longrightarrow E \longrightarrow 0;$$

(2) *we have $\psi(E) = \#(E^G)/(\#E)^d$.*

Proof. To show (1), first note that we can take a set T with trivial G-action when the G-action on E is trivial. We can therefore assume that E is simple. Let N be the normal p-Sylow subgroup of G. We saw in the previous proof that N acts trivially on E, so E is a module for the commutative ring $R = \mathbb{Z}_p[G/N]$. But since G/N is cyclic of order coprime to p, this ring R is a finite product of discrete valuation rings, and in particular, R is a principal ideal ring. Since E is the residue field of a maximal ideal of R, this implies that we can make a short exact sequence as in (1) with $T = G/N$. This proves (1).

To show (2) note first that the canonical map

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}T, \mathbb{Z}T) \longrightarrow \operatorname{Hom}_{\mathbb{Z}_pG}(\mathbb{Z}_pT, \mathbb{Z}_pT)$$

is an isomorphism. Since E is finite, one sees from this and part (1) that there is an exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow \mathbb{Z}T \longrightarrow \mathbb{Z}T \longrightarrow E' \longrightarrow 0,$$

so that E' is finite and $E' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is isomorphic as a $\mathbb{Z}_pG$-module to E. From the definition of $\psi(E)$ one sees that $\operatorname{ord}_p(\psi(E)) = \operatorname{ord}_p(\psi(E'))$. Applying (2.5) we see that $\psi(E') = \#(E'^G)/(\#E')^d$, and (5.2) follows by taking p-parts. □

(5.3) Proposition. *If E is a finite $\mathbb{Z}_pG$-module, and G is cyclic modulo p, but not cyclic, then $\psi(E)$ is an integral multiple of $\#(E^G)/(\#E)^d$.*

Proof. We use induction on the length of E. Note that E^G is a sub-$\mathbb{Z}G$-module of E, and that (5.3) holds for E^G. If $E \neq E^G$ then there is a short exact sequence of $\mathbb{Z}_pG$-modules

$$0 \longrightarrow D \longrightarrow E \longrightarrow S \longrightarrow 0$$

with S simple and E^G contained in the image of D. For $\mathbb{Z}G$-modules M and N and $i \in \mathbb{Z}_{\geq 0}$ let us write $\operatorname{Ext}^i_G(M, N) = M^iN$ as shorthand. Since $S^2V = 0$ by Lemma (5.1) we obtain the following diagram with exact rows

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & S^0V & \longrightarrow & E^0V & \longrightarrow & D^0V & \longrightarrow & S^1V & \longrightarrow & E^1V & \longrightarrow & D^1V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & S^0C & \longrightarrow & E^0C & \longrightarrow & D^0C & \longrightarrow & S^1C & \longrightarrow & E^1C & \longrightarrow & D^1C & \xrightarrow{\delta} & S^2C.
\end{array}$$

The kernels and cokernels of the vertical maps are finite by (4.2), and since the homology groups of the complex of kernels are isomorphic to the homology groups of the complex of cokernels (shifted by 2), it follows with the definition of ψ that

$$\psi(E) = \psi(D)\psi(S)/\#\operatorname{Im}\delta.$$

By induction, (5.3) holds for D and S, so

$$\psi(E) = \frac{\#(D^G)}{(\#D)^d}\,\frac{\#(S^G)}{(\#S)^d}\,\frac{\#\operatorname{Cok}\delta}{\#(S^2C)} = \frac{\#(E^G)}{(\#E)^d}\,\#\operatorname{Cok}\delta.$$

This last equality follows from the fact that $\#E^G = \#D^G$ and that $\#(S^G) = \#(S^2C)$ by (5.1). This proves (5.3). □

Proof of (2.6). If G is cyclic, then $X \cong_G Y$ and the statement follows from (2.5). So assume that G is not cyclic, so that (5.3) applies.

Let E be the cokernel of φ, and let E_p be its p-part. Using (2.4) we see that

$$\operatorname{ord}_p\left(\frac{L(K_X)}{L(K_Y)}\right) = \operatorname{ord}_p\left(\psi(E)\frac{\#E^d}{\#E^G}\right) = \operatorname{ord}_p\left(\psi(E_p)\frac{\#E_p^d}{\#E_p^G}\right) \geq 0,$$

where the inequality follows from (5.3). But switching the role of X and Y gives the opposite inequality, so we have equality.

(5.4) Remark. It follows from the argument above that the map δ in the proof of (5.3) is in fact surjective. We know of no direct way (i.e., no way not using (2.4)) to show this.

6. Conlon induction

In this section we show that (2.6) implies (1.1) by purely representation theoretic arguments. Let G be a finite group, and let $\mathcal{C}$ be a set of representatives of the conjugacy classes of cyclic subgroups of G. By character theory (see [14, Chap. XVIII, Thm. 2.3]) it follows that the isomorphism class of the $\mathbb{Q}G$-module $\mathbb{Q}T$ for any G-set T is determined by the vector $v_T = (\#T^C)_{C\in\mathcal{C}} \in \mathbb{Q}^{\mathcal{C}}$. For each $C \in \mathcal{C}$ the vector $v_{G/C}$ has a non-zero coordinate at $C \in \mathcal{C}$, and all other non-zero coordinates occur at $C' \in \mathcal{C}$ with $\#C' < \#C$. By induction on n it follows that for each $n \in \mathbb{Z}$ the vectors $v_{G/C}$ with $C \in \mathcal{C}$ and $\#C < n$ are linearly independent. Comparing dimensions we see that the vectors $v_{G/C}$ with $C \in \mathcal{C}$ generate $\mathbb{Q}^{\mathcal{C}}$ as a $\mathbb{Q}$-vector space. This implies that for any G-set T there is a positive integer r and two G-sets Z and Z' for which all point stabilizers are cyclic, so that there is an isomorphism of $\mathbb{Q}G$-modules

$$(\mathbb{Q}T)^r \oplus \mathbb{Q}Z \cong \mathbb{Q}Z'.$$

We write this as $rT \cup Z \sim Z'$. This is a special case of Artin's induction theorem [4, §15A].

We will need a p-adic analog of this result known as Conlon's induction theorem [4, §81B]. Conlon's induction theorem says the following. For each G-set T and each prime number p there is a positive integer r and two G-sets Z and Z' for which all point stabilizers are cyclic modulo p, so that there is an isomorphism of $\mathbb{Z}_pG$-modules

$$(\mathbb{Z}_pT)^r \oplus \mathbb{Z}_pZ \cong \mathbb{Z}_pZ'.$$

Let us write this as $rT \cup Z \sim_p Z'$.

We are now ready to prove (1.1) in the general case. It follows from (2.1) that $L(K_X)/L(K_Y)$ is a positive rational number. Let p be a prime number. By Conlon induction there are G-sets T, T_1, T_2 for which all point stabilizers are cyclic modulo p, and there is a positive integer r, so that

$$T_1 \sim_p T \cup rX \sim T \cup rY \sim_p T_2.$$

By applying (2.2) to the outer two linear equivalences, we see that

$$\mathrm{ord}_p(L(K_X)/L(K_Y)) = \frac{1}{r}\,\mathrm{ord}_p(L(K_{T_1})/L(K_{T_2})).$$

For every component F of K_{T_1} or K_{T_2} the group $H = \mathrm{Gal}(K/F)$ is cyclic modulo p. We now apply Artin induction to the extension $F \subset K$ for all these components F: this gives G-sets Z, Z_1, Z_2 for which all point stabilizers are cyclic, and a positive integer s, so that $Z \cup sT_1 \sim Z_1$ and $Z \cup sT_2 \sim Z_2$. What is special about these linear equivalences is that they arise from only applying Artin induction to groups which are cyclic modulo p, and this means by (2.6) that we have

$$s\,\mathrm{ord}_p(L(K_{T_1})/L(K_{T_2})) = \mathrm{ord}_p(L(K_{Z_1})/L(K_{Z_2})).$$

But now Z_1 and Z_2 are linearly equivalent G-sets with cyclic point stabilizers. Since the vectors $v_{G/C}$ above are linearly independent when C ranges over a representative set of conjugacy classes of cyclic subgroups of G, this implies that Z_1 and Z_2 are isomorphic as G-sets. This implies that $L(K_{Z_1}) = L(K_{Z_2})$, so that $\mathrm{ord}_p(L(K_X)/L(K_Y)) = \mathrm{ord}_p(L(K_{T_1})/L(K_{T_2})) = 0$. Since p was arbitrary, it follows that $L(K_X) = L(K_Y)$. This completes the proof of (1.1).

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