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Finite complete intersection algebras and the completeness radical

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Abstract. Let A be a ring, and let B be a finite A-algebra. If B is of the form $A[X_1,\ldots,X_n]/(f_1,\ldots,f_n)$ then we say that B is a complete intersection over A. We show that such an algebra is projective as an A-module and Gorenstein as an A-algebra. Under the condition that A is noetherian we show that the finite A-algebras of the form $A[[X_1,\ldots,X_m]]/(g_1,\ldots,g_m)$ are exactly those complete intersections B over A for which one has $B{=}A{\cdot}1{+}\sqrt{\mathrm{cr}(A){\cdot}B}$. Here $\mathrm{cr}(A)$ denotes the largest ideal of A with respect to which A is complete. This ideal, which we call the completeness radical of A, satisfies the usual radical axioms.

Key words: commutative rings, complete intersection, power series, completeness radical.

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1. Introduction

A finite algebra is a complete intersection if it can be given with the same number of generators and relations. To make this definition precise, one can use polynomial rings or power series rings. The main result of this paper identifies, over a noetherian base ring, the latter type of complete intersections as a subclass of the former; see Theorem (1.3).

Finite complete intersection algebras play a key role in Wiles's recent proof of Fermat's Last Theorem [10, Appendix]. Wiles works in the power series setting, and his base ring is a complete discrete valuation ring. Two basic results in this context are that a finite complete intersection is projective as a module over the base ring, and that it has the Gorenstein property (cf. [7, Lemma 1]). We prove, more generally, that the same results are valid in the polynomial setting, over an arbitrary base ring; see Proposition (1.1).

All rings and algebras in this note are supposed to be commutative with identity element. Let A be a ring, and let B be a A-algebra. Assume that B is a *finite* A-algebra, i. e., that B is finitely generated as an A-module. We say that B is a *complete intersection* over A if there exist a non-negative integer n and $f_1, \ldots, f_n \in A[X_1, \ldots, X_n]$ such that B is isomorphic to $A[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$ as an A-algebra.

For any A-module M let $M^\dagger = \mathrm{Hom}_A(M, A)$ be its A-linear dual. There is a B-module structure on $B^\dagger$ given by $(b\varphi)(x) = \varphi(bx)$ for $b, x \in B$ and $\varphi \in B^\dagger$. Now suppose that B is projective as an A-module. Then we have a natural isomorphism $B \otimes_A B^\dagger \xrightarrow{\sim} \mathrm{End}_A(B)$ sending $b \otimes \varphi$ to the endomorphism $x \mapsto b\varphi(x)$ of B, and we define the *trace* map $\mathrm{Tr}_{B/A} \in B^\dagger$ to be the composite map $B \to \mathrm{End}_A(B) \cong B \otimes_A B^\dagger \to A$; here the first map sends $b \in B$ to the endomorphism $x \mapsto bx$ and the last map is given by $b \otimes \varphi \mapsto \varphi(b)$.

(1.1) Proposition. *Let A be a ring, let n be a non-negative integer, and let $f_1, \ldots, f_n \in A[X_1, \ldots, X_n]$. Assume that $B = A[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$ is a finite A-algebra. Then*

(1) *B is projective as an A-module;*
(2) *$B^\dagger$ is free of rank 1 as a B-module;*
(3) *there is a generator λ of $B^\dagger$ as a B-module such that $\mathrm{Tr}_{B/A} = \det(\partial f_i/\partial X_j)_{ij} \cdot \lambda$.*

In Section 2 below we prove (1), and we indicate how (2) and (3) follow with an argument of Tate. Our main tool is the Koszul complex, which is defined in Section 2. A finite algebra satisfying (1) is *Gorenstein* if $B^\dagger$ is projective of rank 1 as a B-module [6, E.16], so (2) implies that B is Gorenstein over A.

By a *power series complete intersection* over A we mean a finite A-algebra of the form $A[[X_1, \ldots, X_m]]/(g_1, \ldots, g_m)$. We claim that under the condition that A is noetherian, (1.1) remains true when we replace the polynomial ring by a power series ring. To prove this, one can go through the proof of (1.1) and adjust the arguments to power series rings; this is done in [4]. Here we adopt a different approach. For noetherian A we will show, in Section 4, that a finite A-algebra $B = A[[X_1, \ldots, X_m]]/(g_1, \ldots, g_m)$ is in fact a complete intersection over A, so that (1.1) applies. Thus, (1) and (2) of (1.1) hold for B, and a short argument, also given in Section 4, shows that statement (3) with $f_1, \ldots, f_n$ replaced by $g_1, \ldots, g_m$ is also true.

In order to identify *which* complete intersections over a noetherian ring A are power series complete intersections, one is led to consider the *completeness radical* $\mathrm{cr}(A)$ of A, which is defined as the largest A-ideal with respect to which A is complete. We will show in Section 3 that this ideal is well-defined (Theorem (3.1)). It is easy to see that $\mathrm{cr}(A)$ contains the nilradical $\mathrm{nil}(A) = \sqrt{(0)}$ of A, and that it is contained in the Jacobson radical. Furthermore, it has the following properties, which are standard requirements in theories of radicals [5].

(1.2) Proposition. *If A is a noetherian ring and $\mathfrak{a}$ is an A-ideal, then we have*

$$(1) \qquad \mathrm{cr}(A) = \sqrt{\mathrm{cr}(A)};$$
$$(2) \qquad \mathfrak{a} \subset \mathrm{cr}(A) \ \Rightarrow\ \mathrm{cr}(A/\mathfrak{a}) = \mathrm{cr}(A)/\mathfrak{a};$$
$$(3) \qquad \mathrm{cr}(A/\mathfrak{a}) = 0 \ \Rightarrow\ \mathrm{cr}(A) \subset \mathfrak{a}.$$

This result is proved in Section 3. Note that (2) implies that $\mathrm{cr}(A/\mathrm{cr}(A)) = 0$.

Our main result, which we now formulate, identifies the class of power series complete intersections over a noetherian ring A as a certain subclass of the class of complete intersections over A. For any ring R we let $R_{\text{red}} = R/\text{nil}(R)$.

(1.3) Theorem. *Let A be a noetherian ring and let B be a finite A-algebra. Then B is a power series complete intersection over A if and only if these two conditions hold:*

(1) *B is a complete intersection over A;*
(2) *the structure homomorphism $A \to (B/\text{cr}(A)B)_{\text{red}}$ is surjective.*

The proof of this theorem is given in Section 4.

If (1) holds, with A connected and $B \neq 0$, then it follows with (1.1)(1) and (1.2)(1) that the map in (2) has kernel $\text{cr}(A)$. For example, take $A = \mathbb{Z}$, so that $\text{cr}(A) = 0$. Then we conclude that a non-zero power series complete intersection B over $\mathbb{Z}$ is the same as a complete intersection satisfying $B_{\text{red}} = \mathbb{Z}$. As an example of a different nature, consider the case that A is a complete noetherian local ring. Then $\text{cr}(A)$ is the maximal ideal of A, and (2) is equivalent to the condition that B, if non-zero, is local with the same residue field as A.

2. Ring-theoretic properties of complete intersections

In this section we prove (1.1). We first recall the definition of the Koszul complex.

Let R be a ring and let M be an R-module. Let $V = R^n$ for a positive integer n, and let $f = (f_1, f_2, \ldots, f_n) \in V$. We define the Koszul complex

$$K(f, M): \qquad 0 \to K_n(f, M) \xrightarrow{d} \cdots \xrightarrow{d} K_1(f, M) \xrightarrow{d} K_0(f, M) \to 0$$

as follows: we take $K_k(f, M) = \text{Hom}_R(\bigwedge_R^k V, M)$ and for $\varphi \in K_k(f, M)$ we define $d\varphi \in K_{k-1}(f, M)$ by $(d\varphi)(x) = \varphi(f \wedge x)$. The kth homology group of this complex is denoted by $H_k(f, M)$. Let I be the R-ideal generated by the f_i. Then we have a canonical isomorphism $H_0(f, M) \cong M/IM$.

We will use in the sequel that each $H_k(f, M)$ is annihilated by I. To see this, let $\varphi \in K_k(f, M)$ satisfy $d\varphi = 0$. For each generator f_i of I we need to produce an element $\psi \in K_{k+1}(f, M)$ with $d\psi = f_i \varphi$. Write $V = Re_i \oplus V'$, where e_i is the ith standard basis vector of V over R, and where V' is spanned by the other standard basis vectors. We can write any $x \in \bigwedge^{k+1} V$ as $x = e_i \wedge x' + x''$ for unique $x' \in \bigwedge^k V'$ and $x'' \in \bigwedge^{k+1} V'$. Now define ψ by $\psi(x) = \varphi(x')$. One deduces from $d\varphi = 0$ that $d\psi = f_i \varphi$.

We need one lemma about Koszul complexes, which can also be found in [3, Theorem 1.6.16]. We will say that a sequence $p_1, p_2, \ldots, p_n$ in R is *weakly M-regular* if for all i with $1 \leq i \leq n$ multiplication by p_i on the R-module $M/(p_1, \ldots, p_{i-1})M$ is an injection.

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(2.1) Lemma. *Let $f_1, \ldots, f_n$ be a sequence in a ring R, and let M be an R-module. If the R-ideal $I = (f_1, \ldots, f_n)$ contains a weakly M-regular sequence of length n, then $K(f, M)$ is exact in positive dimension, i. e., $H_i(f, M) = 0$ for $i > 0$.*

Proof. Suppose $p_1, \ldots, p_n$ is a weakly M-regular sequence in I. We prove for j with $0 \leq j \leq n$ and all $i > j$ that $H_i(f, M/(p_1, \ldots, p_j)M) = 0$. For $j = n$ this is trivial, and for $j = 0$ this is our lemma. We proceed by induction, decreasing j by 1 in each step.

Put $M' = M/(p_1, \ldots, p_{j-1})M$. By our regularity assumption we have a short exact sequence of R-modules

$$0 \to M' \xrightarrow{p_j} M' \to M'/p_j M' \to 0.$$

For each k we apply the functor $\mathrm{Hom}_R(\bigwedge^k V, -)$, which is exact because $\bigwedge^k V$ is a free R-module. Thus we get a short exact sequence of complexes

$$0 \to K(f, M') \xrightarrow{p_j} K(f, M') \to K(f, M'/p_j M') \to 0.$$

Since the homology groups of $K(f, M')$ are annihilated by I and in particular by p_j, the long exact sequence of homology groups gives us short exact sequences

$$0 \to H_i(f, M') \to H_i(f, M'/p_j M') \to H_{i-1}(f, M') \to 0$$

for $i \geq 0$. The induction hypothesis implies that the middle group is zero for $i \geq j$, and we deduce that $H_i(f, M') = 0$ for $i \geq j - 1$. This completes the induction step, and proves the lemma. □

We now prove part (1) of (1.1). Denote the polynomial ring $A[X_1, \ldots, X_n]$ by R. Note that B is integral over A, so that the image α_i of X_i in B satisfies a relation $p_i(\alpha_i) = 0$ with $p_i \in A[X_i]$ monic. The sequence $p_1, \ldots, p_n$ lies in $(f_1, \ldots, f_n)$ and it is weakly R-regular, i.e., we have short exact sequences

$$0 \to R/(p_1, \ldots, p_{i-1}) \xrightarrow{p_i} R/(p_1, \ldots, p_{i-1}) \to R/(p_1, \ldots, p_i) \to 0$$

for $i = 1, 2, \ldots, n$. Note that the modules in this sequence are free (but not necessarily finite) over A. Hence exactness is preserved when we tensor over A with any A-module M, and we deduce that $p_1, \ldots, p_n$ is weakly $(R \otimes_A M)$-regular. By (2.1) this implies that $H_i(f, R \otimes_A M) = 0$ if $i > 0$.

Let us now show that B is flat as an A-module by proving that the functor $B \otimes_A -$ is exact. Suppose we have a short exact sequence of A-modules

$$0 \to M' \to M \to M'' \to 0.$$

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Since R is free over A, tensoring with R over A preserves exactness. Subsequent application of the exact functors $\mathrm{Hom}_R(\bigwedge^k V, -)$ then gives a short exact sequence of complexes

$$0 \to K(f, R \otimes_A M') \to K(f, R \otimes_A M) \to K(f, R \otimes_A M'') \to 0.$$

Since $H_0(f, N) = N/IN = B \otimes_R N$ for any R-module N, the long exact sequence of homology groups gives an exact sequence

$$H_1(f, R \otimes_A M'') \to B \otimes_A M' \to B \otimes_A M \to B \otimes_A M'' \to 0.$$

We already saw that the H_1 vanishes, so indeed the functor $B \otimes_A -$ is exact, and B is flat over A. (Alternatively, one can deduce this from the fact that $K(f, R)$ is an A-flat resolution of B, so that $\mathrm{Tor}^i_A(B, M) = H_i(f, R \otimes_A M) = 0$ for $i > 0$.)

Since finitely presented flat modules are projective [8, §7], it remains to show that B is finitely presented as an A-module. To see this consider the surjection of the finitely generated free A-module $R_0 = R/(p_1, \ldots, p_n)$ to B. Its kernel W is finitely generated over R_0, and since R_0 is finite over A this implies that W is finitely generated as an A-module. Thus B is finitely presented over A. This proves part (1) of (1.1).

Next, we quote a theorem of Tate, which given (1) in (1.1) will show (2) and (3).

(2.2) Theorem (Tate). *Let A be a ring, let R be an A-algebra, and let $I = (f_1, \ldots, f_n)$ be an R-ideal. Suppose that the quotient ring $B = R/I$ is finitely generated and projective as an A-module. Suppose further that the kernel of the composite ring homomorphism $\psi\colon S = B \otimes_A R \to B \otimes_A B \to B$ is as an S-ideal generated by n elements $g_1, \ldots, g_n$, and that $H_i(f, R) = H_i(g, S) = 0$ for all $i > 0$. Write $1 \otimes f_i = \sum_j b_{ij} g_j$ with $b_{ij} \in S$ and let $d = \det(b_{ij}) \in S$. Define the map $t\colon B^\dagger \to B$ by $t(\varphi) = (\varphi'(d) \bmod I)$, where $\varphi'\colon S \to R$ is the R-linear map $\varphi \otimes 1$. Then t is a B-linear isomorphism and $t(\mathrm{Tr}_{B/A}) = \psi(d)$.*

For the proof of (2.2) we refer to [9, Appendix A]. It is assumed there that B is free over A. One reduces to this case by noting that (2.2) can be verified locally on A. The regularity assumption in [9, (A.3)] is only used to show that the Koszul complexes satisfy the hypothesis of (2.2).

Using that $H_1(g, S) = 0$ one can show that the isomorphism t in (2.2) does not depend on the choice of the b_{ij}, for given f_i and g_j.

To show (2) and (3) of (1.1), we apply (2.2) with $R = A[X_1, \ldots, X_n]$ and $g_i = 1 \otimes X_i - \alpha_i \otimes 1 \in S = B \otimes_A R$, where α_i is the image of X_i in B. We already saw in the proof of part (1) of (1.1) that $H_i(f, R) = 0$ for $i > 0$. We have $\mathrm{Ker}\,\psi = (g_1, \ldots, g_n)$ and since the sequence $g_1, \ldots, g_n$ is weakly S-regular, we also have $H_i(g, S) = 0$ for $i > 0$.

Thus, the conditions of (2.2) are met, and (2) follows. To see that (3) holds one checks that $\psi(b_{ij}) = (\partial f_i/\partial X_j \bmod I)$.

Note that the argument above provides us with an explicit B-module isomorphism $B^\dagger \to B$, but that the isomorphism depends on the chosen presentation of B as an A-algebra with generators α_i and relations f_i.

3. The completeness radical

In this section we show that the completeness radical $\mathrm{cr}(A)$ is well-defined, we prove (1.2), and we determine the behavior of the completeness radical under finite flat extensions.

Throughout this section, A is a noetherian ring, M is a finitely generated A-module, and $\mathfrak{a}$ and $\mathfrak{b}$ are A-ideals. The $\mathfrak{a}$-adic completion $\hat{M}$ of M is defined to be the projective limit $\hat{M} = \varprojlim M/\mathfrak{a}^n M$. There is a canonical map $M \to \hat{M}$, and M is said to be $\mathfrak{a}$-complete if this map is an isomorphism. Since M is finitely generated and A is noetherian, the completion $\hat{M}$ is canonically isomorphic to $M \otimes_A \hat{A}$ (see [1, 10.13]). Completion of finitely generated A-modules preserves exactness [1, 10.12].

(3.1) Theorem. *Let A be a noetherian ring, and let M be a finitely generated A-module. Then there is a unique A-ideal $\mathfrak{c}$ with the following property: if $\mathfrak{a}$ is an A-ideal, then M is $\mathfrak{a}$-complete if and only if $\mathfrak{a}$ is contained in $\mathfrak{c}$.*

In the case $M = A$ we write $\mathrm{cr}(A)$ for the ideal $\mathfrak{c}$ from Theorem (3.1), and we call it the *completeness radical* of A. We begin the proof of (3.1) with a few lemmas.

(3.2) Lemma. *Suppose $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules. Then M is $\mathfrak{a}$-complete if and only if both M' and M'' are $\mathfrak{a}$-complete.*

Proof. Since A is noetherian and M is finitely generated, M' and M'' are also finitely generated. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & \hat{M}' & \longrightarrow & \hat{M} & \longrightarrow & \hat{M}'' & \longrightarrow & 0. \end{array}$$

By the snake lemma, f is an isomorphism if f' and f'' are isomorphisms. Conversely, suppose that f is an isomorphism. Then $\operatorname{Ker} f'$ and $\operatorname{Coker} f''$ are zero, and $\operatorname{Ker} f'' \cong \operatorname{Coker} f'$. Every element y of $\operatorname{Ker} f''$ is annihilated by some element $1 + x$ with $x \in \mathfrak{a}$ (see [1, 10.17]), and since $\operatorname{Ker} f''$ is finitely generated one can choose x independently of y. But $1 + x$ acts as a unit on every $\hat{A}$-module [1, 10.15 (iv)], and in particular on $\hat{M}'$, so $1 + x$ acts surjectively on $\operatorname{Coker} f'$. This shows that $\operatorname{Ker} f'' = \operatorname{Coker} f' = 0$. □

(3.3) Lemma. *The A-module M is $\mathfrak{a}$-complete if and only if there exists an $\hat{A}$-module structure on M that is compatible with its A-module structure.*

Proof. Note that $\hat{M}$ is a module over $\hat{A}$ and that the map $M \to \hat{M}$ is A-linear. This shows "only if". For the "if" part, suppose that M has a compatible $\hat{A}$-module structure. Since $\hat{A}$ is noetherian [1, 10.27], there is an exact sequence

$$0 \to K \to \hat{A}^n \to M \to 0$$

of finitely generated $\hat{A}$-modules. But $\hat{A}$ is complete at $\mathfrak{a}\hat{A}$ (see the proof of [1, 10.15]), so by (3.2), now applied to $\hat{A}$-modules, M is $\mathfrak{a}\hat{A}$-complete, which is the same as being $\mathfrak{a}$-complete. □

(3.4) Lemma. *If M is $\mathfrak{a}$-complete, and $M/\mathfrak{a}M$ is $\mathfrak{b}$-complete, then M is $\mathfrak{b}$-complete.*

Proof. We first show by induction that $M/\mathfrak{a}^n M$ is $\mathfrak{b}$-complete for all $n \geq 0$. For $n = 0$ this is trivial. To make the induction step suppose that $n \geq 1$ and consider the exact sequence of A-modules

$$0 \to \mathfrak{a}^{n-1}M/\mathfrak{a}^n M \to M/\mathfrak{a}^n M \to M/\mathfrak{a}^{n-1}M \to 0.$$

If $\mathfrak{a}^{n-1}$ is generated as an A-ideal by $a_1, \ldots, a_k$, then there is a surjection $(M/\mathfrak{a}M)^k \to \mathfrak{a}^{n-1}M/\mathfrak{a}^n M$ sending $(m_i)_{i=1}^k$ to $\sum_i a_i m_i$. With (3.2) one sees that $\mathfrak{a}^{n-1}M/\mathfrak{a}^n M$ is $\mathfrak{b}$-complete, and by applying (3.2) to the sequence above we can make the induction step.

By (3.3) we know that $M/\mathfrak{a}^n M$ has a compatible module structure over the $\mathfrak{b}$-completion of A. Then the projective limit of $M/\mathfrak{a}^n M$ has such a structure too. But this projective limit is M so M is $\mathfrak{b}$-complete. □

The special case $M = A$ of the following lemma can be found in [8, §8, Exercises 1, 2]; the "if"-part can be deduced from [1, Chapter 10, Exercise 5].

(3.5) Lemma. *The A-module M is $(\mathfrak{a} + \mathfrak{b})$-complete if and only if it is both $\mathfrak{a}$-complete and $\mathfrak{b}$-complete.*

Proof. Suppose that M is $\mathfrak{a}$-complete and $\mathfrak{b}$-complete. Then (3.2) implies that $M/\mathfrak{a}M$ is also $\mathfrak{b}$-complete, and therefore $(\mathfrak{a} + \mathfrak{b})$-complete. By (3.4) we then see that M is $(\mathfrak{a} + \mathfrak{b})$-complete.

Conversely, let $\tilde{A}$ be the $(\mathfrak{a} + \mathfrak{b})$-completion of A and let $\hat{A}$ be the $\mathfrak{a}$-completion of A. We have canonical maps $A \to \hat{A} \to \tilde{A}$, so a compatible $\tilde{A}$-module structure induces a compatible $\hat{A}$-module structure on M. If M is $(\mathfrak{a} + \mathfrak{b})$-complete then one deduces with (3.3) that it is $\mathfrak{a}$-complete, and by symmetry it is $\mathfrak{b}$-complete as well. This shows (3.5). □

Proof of (3.1). Take $\mathfrak{c}$ to be the sum of all A-ideals $\mathfrak{a}$ for which M is $\mathfrak{a}$-complete. It follows from (3.5) that $\mathfrak{c}$ has the stated property, since A is noetherian and M is (0)-complete. □

Proof of (1.2). To see (1), note that for any $k \geq 1$ completeness at $\mathfrak{a}^k$ is the same as completeness at $\mathfrak{a}$. For $\mathfrak{a} \subset \mathrm{cr}(A)$ we see from (3.2) and (3.4) that $A/\mathfrak{a}$ is a $\mathfrak{b}$-complete A-module if and only if A is $\mathfrak{b}$-complete, and this implies property (2) of (1.2). To see (3), one infers from (3.2) that $A/\mathfrak{a}$ is a $\mathrm{cr}(A)$-complete A-module for all A-ideals $\mathfrak{a}$, so that $\mathrm{cr}(A/\mathfrak{a}) \supset (\mathrm{cr}(A) + \mathfrak{a})/\mathfrak{a}$. This shows (1.2). □

(3.6) Remark. The ideal $\mathrm{cr}(A)$ also occurs in Brewer [2, Theorems 50 and 59]. He writes $I_c(A)$ for the set of elements x in A for which there is an A-algebra homomorphism $A[[X]] \to A$ sending X to x. Let us show that $I_c(A) = \mathrm{cr}(A)$ for noetherian A. Suppose that an A-algebra homomorphism $A[[X]] \to A$ sends X to x. Applying (3.2) to the $A[[X]]$-modules $M = A[[X]]$ and $M'' = A$, with $\mathfrak{a} = (X)$, we see that A is (x)-complete, so $(x) \subset \mathrm{cr}(A)$. Conversely, if $x \in \mathrm{cr}(A)$ then A is (x)-complete. Completing the A-algebra homomorphism $A[X] \to A$ that maps X to x at the ideal (X) one then gets a map $A[[X]] \to A$ sending X to x, so that $x \in I_c(A)$.

(3.7) Proposition. *Let A be a noetherian ring and let B be a finite A-algebra that is projective as an A-module. Then we have $\mathrm{cr}(B) = \sqrt{\mathrm{cr}(A)B}$.*

The proof depends on a lemma. Let, as before, M be a finitely generated A-module, and let b be an A-endomorphism of M. We say that M is *b-complete* if the natural map $M \to \varprojlim M/b^n M$ is an isomorphism; or, equivalently, if it is (X)-complete as an $A[X]$-module, with X acting as b.

(3.8) Lemma. *Suppose that $\mathrm{cr}(A) = 0$ and that M can be embedded in a free A-module. If b is an A-endomorphism of M, then M is b-complete if and only if $b^n = 0$ for some $n \geq 0$.*

Proof. The “if” part is clear. For the “only if” part, we first show that if M is (a)-complete, with $a \in A$, then $aM = 0$. Suppose that M is (a)-complete, and let $f\colon M \to A$ be any A-linear map. Then the A-ideal $I = f(M)$ is (a)-complete by (3.2), so A/I and I are both aI-complete. Hence by (3.2) the ring A is aI-complete, and therefore $aI \subset \mathrm{cr}(A) = 0$. Embedding M in a free A-module, we see that for any $m \in M$ all coordinates of am are zero. Therefore $aM = 0$, as required.

Now assume that M is b-complete. Since M is finitely generated over A, the endomorphism b satisfies a relation of the form $b^n + a_1 b^{n-1} + \cdots + a_n = 0$, with $a_1, \ldots, a_n \in A$. We prove by induction on n that we then have $b^n = 0$. For $n = 0$ this is clear. Next let $n > 0$, and put $c = b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1}$. Then $cb + a_n = 0$, so M is $cb + a_n$-complete. Since by hypothesis it is b-complete, we see from (3.5) (applied with $A[X]$ as the base ring) that it is a_n-complete. By what we just proved, this implies that $a_n M = 0$. Therefore cb annihilates M, and c annihilates bM. Hence the restriction of b to bM satisfies a monic equation of degree $n-1$. Since bM can be embedded in a free A-module, the induction hypothesis implies that b^{n-1} annihilates bM, so $b^n M = 0$. □

The condition on M in (3.8) is equivalent to the condition that the canonical map $M \to M^{\dagger\dagger}$ be injective.

Proof of (3.7). First assume that $\mathrm{cr}(A) = 0$. Since B is projective, it is a direct summand of a free A-module. An element $b \in B$ belongs to $\mathrm{cr}(B)$ if and only if B is complete with respect to the A-endomorphism $x \mapsto bx$. By (3.8), this occurs if and only if b is nilpotent. Hence $\mathrm{cr}(B) = \sqrt{(0)} = \sqrt{\mathrm{cr}(A)B}$, as required.

In general we pass to the ring $A' = A/\mathrm{cr}(A)$, which by (2) of (1.2) satisfies $\mathrm{cr}(A') = 0$. By the case just treated, the A'-algebra $B' = B \otimes_A A' = B/\mathrm{cr}(A)B$ has $\mathrm{cr}(B') = \sqrt{(0)}$. Also, (3.2) implies that B is $\mathrm{cr}(A)$-complete as an A-module, so $\mathrm{cr}(A)B \subset \mathrm{cr}(B)$, and (3) of (1.2) gives $\mathrm{cr}(B') = \mathrm{cr}(B)/\mathrm{cr}(A)B$. Thus we have $\mathrm{cr}(B)/\mathrm{cr}(A)B = \sqrt{(0)}$, as required. □

The proposition may break down for general finite algebras, even when they are assumed to be faithful and connected. This is shown by the following example. Take $A = \mathbb{Z}_p[X]$, where p is a prime number and $\mathbb{Z}_p$ denotes the ring of p-adic integers, and put $B = \{(f, x) \in A \times \mathbb{Z}_p : f(0) \equiv x \bmod p\mathbb{Z}_p\}$. Let the A-algebra structure map $A \to B$ be given by $f \mapsto (f, f(0))$. Then B is reduced and $\mathrm{cr}(B) = \{0\} \times p\mathbb{Z}_p$, while $\mathrm{cr}(A) = 0$.

4. Power series complete intersections

In this section we show (1.3). Let $R = A[X_1, \ldots, X_n]$ and let $\mathfrak{a}$ be the R-ideal $(X_1, \ldots, X_n)$. Then the $\mathfrak{a}$-completion $\hat{R}$ is the ring $A[[X_1, \ldots, X_n]]$.

We first prove "if". Suppose that $B = R/(f_1, \ldots, f_n)$ is a finite A-algebra. Denote the image of X_i in B by α_i. Assuming (2) of (1.3) we perform a translation of the X_i by elements of A so that $\alpha_i \in \sqrt{\mathrm{cr}(A)B}$. Any finitely generated A-module is $\mathrm{cr}(A)$-complete by (3.2), so $\mathrm{cr}(B) \supset \mathrm{cr}(A)B$. Since $\mathrm{cr}(B)$ is a radical ideal we see that $\alpha_i \in \mathrm{cr}(B)$ and that B is complete at $(\alpha_1, \ldots, \alpha_n)$. Now consider the exact sequence of R-modules

$$R^n \xrightarrow{(f_i)} R \longrightarrow B \longrightarrow 0.$$

Since R is noetherian, completion at $\mathfrak{a}$ preserves exactness. But B is $\mathfrak{a}$-complete, so $B = \hat{R}/(f_1, \ldots, f_n)$. This proves the "if" part of (1.3). Note that we showed that a complete intersection satisfying condition (2) can be written as a power series complete intersection with the *same* number of generators.

We now show "only if". Suppose that $f_1, \ldots, f_n \in \hat{R}$ are such that $B = \hat{R}/(f_1, \ldots, f_n)$ is a finite A-algebra. We know that R is noetherian and that B is finitely generated over R, so it follows from (3.3) that B is an $\mathfrak{a}$-complete R-module. By exactness of completion, the cokernel of the natural map $\pi\colon R \to B$ has trivial $\mathfrak{a}$-completion. But by (3.2) this cokernel is also complete, so that π is surjective. Put $J = \operatorname{Ker}\pi$. The $\mathfrak{a}$-completion $\hat{J}$ of J is the $\hat{R}$-ideal $(f_1, \ldots, f_n)$, so $J/\mathfrak{a}J$ can be generated as an A-module by the images of n elements $g_1, \ldots, g_n \in J$. Since $\mathfrak{a}\hat{R}$ lies in the Jacobson radical of $\hat{R}$, Nakayama's lemma implies that $(g_1, \ldots, g_n)\hat{R} = (f_1, \ldots, f_n)$. Let $D = R/(g_1, \ldots, g_n)$. Completing the canonical map $D \xrightarrow{\varphi} B$ at $\mathfrak{a}$ we get an isomorphism $\hat{D} \to B$, so the D-ideal $H = J/(g_1, \ldots, g_n) = \operatorname{Ker}\varphi$ satisfies $\hat{H} = 0$. We now have $H/H^2 = H \otimes_D (D/H) = H \otimes_D \hat{D} = \hat{H} = 0$, so $H = H^2$. By the "determinant trick" [1, 2.5] this implies that $(1-e)H = 0$ for some $e \in H$, so $H = eD$ and $e^2 = e$. But then $D[X]/((1-e)X - e) \cong D/H = B$, so by lifting e to an element $\tilde{e} \in R$ we have $B = A[X_1, \ldots, X_{n+1}]/(g_1, \ldots, g_{n+1})$, where $g_{n+1} = (1-\tilde{e})X_{n+1} - \tilde{e}$. This shows statement (1) of (1.3).

Choosing $g_i \in f_i + \mathfrak{a}\hat{J}$, for $1 \le i \le n$, one sees that the image of the $(n+1)\times(n+1)$-determinant $\det(\partial g_i/\partial X_j)$ in B lies in $1 + \mathfrak{a}B$ times the image of the $n \times n$-determinant $\det(\partial f_i/\partial X_j)$ in B. Since $1 + \mathfrak{a}B$ consists of units in B, this shows our claim in the introduction that the polynomial ring in (1.1) can be replaced by the power series ring if A is noetherian.

It remains to show that statement (2) of (1.3) holds. We know that B is $\mathfrak{a}$-complete so the images of the X_i in B lie in $\operatorname{cr}(B)$. This implies that the structure homomorphism $A \to B/\operatorname{cr}(B)$ is surjective. We already know that B is a complete intersection over A, so by (1.1) we see that B is projective as an A-module. Statement (2) of (1.3) now follows from (3.7). □

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