The Droste-effect and the exponential transform

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Abstract

In this note we give a correspondence between three kinds of pictures. First, we consider pictures with the "Droste-effect": a scaling symmetry. These are drawn on a plane with a special point, which is the center of the scaling symmetry. Then we use the complex exponential map to transform these pictures to doubly periodic pictures, commonly known as wallpaper pictures, which are drawn on the entire plane. By rolling up the plane according to the periods, we get pictures drawn on a compact Riemann surfaces of genus 1: donut surfaces. As an application, we show the how the notion of a "Dehn twist" on a donut gives rise to a continuous interpolation between the straight world and the curved world of Escher's 1956 lithograph *Print Gallery*.

1. Introduction

This lecture grew out of the project *Escher and the Droste-effect* [1, 2] that uncovered the mathematical structure of Escher's ingenious lithograph *Print Gallery*. This lithograph can be understood with the concepts of Droste-effect, conformal transformations, pull-back and push-forward of images, and the complex exponential map. While the insight a mathematician may have in these concepts benefits greatly from knowledge of complex numbers, we intend to show that the basic notions and the operations on images that they lead to, can be explained without any mathematical prerequisites. In order to stress the general nature of these operations we will illustrate this with different graphic material rather than the well-known and well-published sketches [2, 3, 5] that Escher made for his lithograph. Escher's lithograph is based on sketches of a man looking at a print of a scene that he himself is a part of. We use a different self-referential image: a cruise ship[1] with a swimming pool that floats in its own pool. We will show how this initial picture can be transformed to a twisted version, exactly like Escher's *Print Gallery*. The transformation is given in two steps: a *pull-back* to a wallpaper pattern, and then a *push-forward*. The terminology and the precise mappings are explained in sections 4 and 5. In the last section, we will show how the mathematical notion of a Dehn twist on a Riemann surface gives rise to a continuous animation that interpolates between the normal and the twisted version.

[1] The pictures of the ship were kindly made available by Mike O'Brien (www.abergwaun.com).

2. Droste-effect

The dictionary definition [6] of the Dutch word *Droste-effect* translates as *repeating visual effect that occurs when an image contains a copy of that image which in turn etc.* It is named after the metal tins of cocoa of the Dutch chocolate maker *Droste*. We refer to an image with this effect as a Droste picture.

Let us first explain what properties or characteristics of a Droste picture we will consider as *intrinsic*. By intrinsic we mean that this property should not change when we change the image in a way that is of no consequence for the Droste-effect, such as giving the nurse in Figure 1 a different hat.

Figure 1: *The Droste-effect.*

First, a Droste picture has a *center*. To find the center, take any point in the image, for instance the nose of the nurse in Figure 1, locate that nose on the image-within-the-image, locate the next smaller nose, and keep going. The sequence of noses obtained in this way converges to a limit point that is independent of the point that we start with. This limit point is the center of the Droste picture.

The second intrinsic property is the *scaling factor*. It can be measured as the ratio between the size of an object in the image, say the vertical size of the nurse, and the size of the same object on the image-within-the-image. For Figure 1 this factor is about 6.5.

The border of a Droste picture is not intrinsic. In fact, any Droste picture within certain borders can extended to the entire plane. To see this, one magnifies the image-within-the-image by the scaling factor, and repeats. Thus, one can let the picture grow outwards, and every point of the plane is covered this way. From now on, our Droste pictures will be drawn on the entire plane with one marked point: the center.

Suppose that a picture of an actual three-dimensional scenery is a Droste picture. Then one can wonder whether this scenery necessarily contains a two-dimensional print, picture, or other representation of the same scenery. For instance, in Figure 1 we think of the nurse as being three-dimensional, holding a two-dimensional image on the cocoa box, until we zoom in further to the next level. In Escher's sketches for *Print Gallery* of a man looking at a print of a scene that contains himself, this is the case too. But our next example will be different in this respect. Thus, boundaries in the Droste picture, such as the rim of the cocoa package on the nurse's tray, where one would say that the nesting depth increases by 1, should not be considered intrinsic.

In the end, only the scaling factor and the center remain as "intrinsic properties". Thus, our mathematical definition of a picture with the Droste-effect is that it is a picture drawn on the entire plane with one marked point, which is scale invariant under scaling with respect to the marked point by a factor which is larger than 1.

When the scaling factor of a Droste picture is so large that the image-within-the-image can no longer be detected because of limitations of the resolution of computer screen and printer or of the human eye, then we can still make the Droste-effect visible with an animation that zooms in on the center. Alternatively, one can show a number of frames of such an animation. This is what Escher did in his four straight sketches for *Print Gallery*, which together yield a Droste picture with a scaling factor of 256; see [2].

We illustrate this here with another Droste picture in Figure 2. On the top we see a cruise ship, and we zoom in on the sun deck in the midsection of the ship. A closer look reveals the swimming pool on the deck, with a passenger and a toy boat floating in it. But the toy boat is an exact copy of the ship, so zooming in further takes us to the top picture again. Completing the Droste picture outside the border of the original picture reveals that the ship we started with, is not on the ocean, but it is in the swimming pool on the deck of an even bigger ship. The scaling factor of this Droste picture is about 227.

Figure 2: *Cruise ship with a pool.*

3. Conformal transformations

There are many well-known transformations of images: pictures can be for instance be translated, scaled or rotated. These operations all preserve straight lines, angles, and ratios of distances. They also preserve orientation, i.e., the notion at each point of which way is turning clockwise, and which way is counterclockwise. Reflecting in a line also preserves straight lines, and distances, but it flips orientation.

Conformal transformations are defined by a weaker property: they only need to preserve angles and orientation. That means that on a very small scale, or, mathematically more precisely, on an *infinitesimal* scale, such transformations are given by translating, rotating and scaling. Thus, small shapes are not distorted, and only larger forms can be bent or distorted. This property makes conformal maps especially suitable for manipulating images because details are preserved.

One way to specify a conformal transformation is by formulas with complex numbers. Both classical and modern mathematics provide an ample supply of conformal transformations. It can be quite interesting to see what the visual effect is of well-known conformal functions on well-known pictures. In Figure 3 the Riemann ζ-function is used to transform *Café terrace at night* (1888) of Vincent van Gogh.

Figure 3: *Van Gogh meets Riemann.*

Instead of giving formulas, one can identify a transformation by showing what happens when it is applied to a standard grid consisting of small squares. This tool was employed frequently by Escher. Whether or not a transformation is conformal is easy to spot on the transformed grid: conformality means that sufficiently small squares in the transformed grid are very close to being squares themselves. When Escher was constructing the grid for *Print Gallery* he took care that the grid paper he made for his transformation satisfied this condition; see [3].

Conformal transformations have amazing mathematical properties. They are very rigid, in the sense that if you know what a conformal transformation does on a small section of the image, then there can be no more than one way to continue this transformation. In practice, one can approximate this continuation by seeing how to complete the grid in such a way that small squares stay small squares.

For instance, if we start with a circle in the plane, then we can consider the transformation that wraps the circle around itself twice. It turns out that we can make this into a conformal map on the entire plane without the center of the circle. Moreover, there is a unique way to do this. Viewing the plane as the plane of complex numbers, with the given circle as the unit circle, this transformation is given by *squaring* these complex numbers.

4. Pull-back and push-forward

The transformations of scaling and rotating are invertible: we can scale or rotate back. But the transformations we will need, and the "squaring map" given by wrapping a circle around itself twice, is not invertible, because two distinct points can be mapped to the same image point. This implies that we have to be more precise about what it means to transform an image according to such a transformation. It turns out that we have two notions: pull-back and push-forward. Suppose we are given two spaces X and Y; say both are planes on which we will draw pictures. A *map* f from X to Y is a way to associate to each point P in X an *image* point $f(P)$ in Y. We will see that the pull-back over f transforms any picture on Y into a picture on X. The push-forward through f transforms pictures on X to pictures on Y, but it does not accept all pictures on X as input.

For example, let X and Y be planes with a circle in each. We now consider a map that wraps the circle on X twice around the circle in Y, and extend this to a conformal map from X to Y. Recall that this extension can be made in a unique way. We will write $f(z) = z^2$ for the image in Y of a point z in X. In Figure 4 the plane X is on the left, and Y is on the right, and the circles run along the inside of the handle.

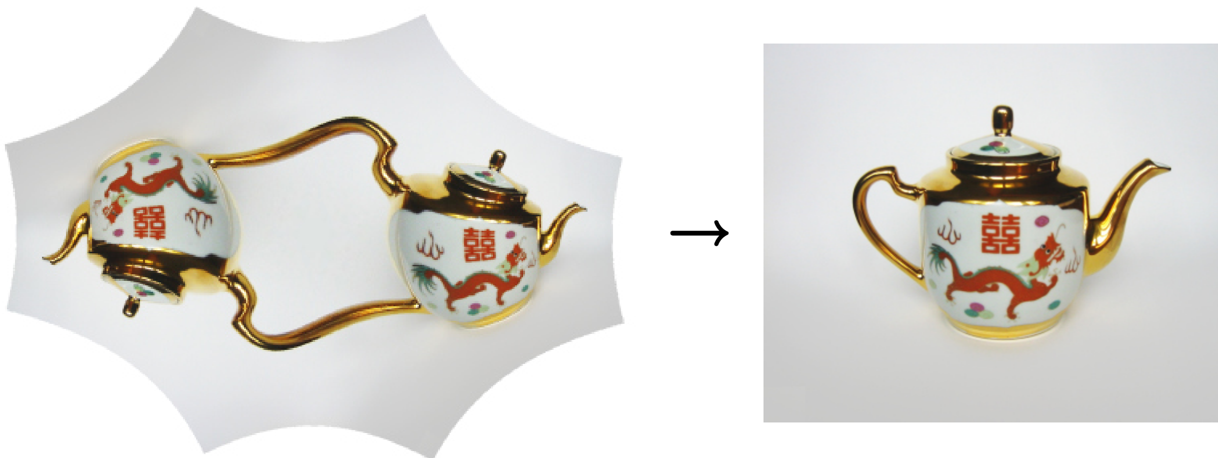

Figure 4: *Pull back of a tea pot. The arrow indicates the map* $z \mapsto z^2$.

We now start with the picture of the tea pot on the plane Y on the right. In order to produce the pull-back picture on the plane X on the left, we color each point z on X by computing its image z^2 in Y and looking up the color at z^2 of the picture on Y. If we write $-z$ for the reflection of z in the center of the circle on X, then z and $-z$ have the same image in Y. This implies that the tea pot, which has no symmetry at all, has a pull-back which has a point reflection symmetry at the center.

If we start with the picture on the left, we can also reconstruct the picture on the right as the *push-forward* through the squaring map. However, not all pictures can be pushed forward through this map: the picture must have the point reflection symmetry. Pushing the picture forward then removes this symmetry.

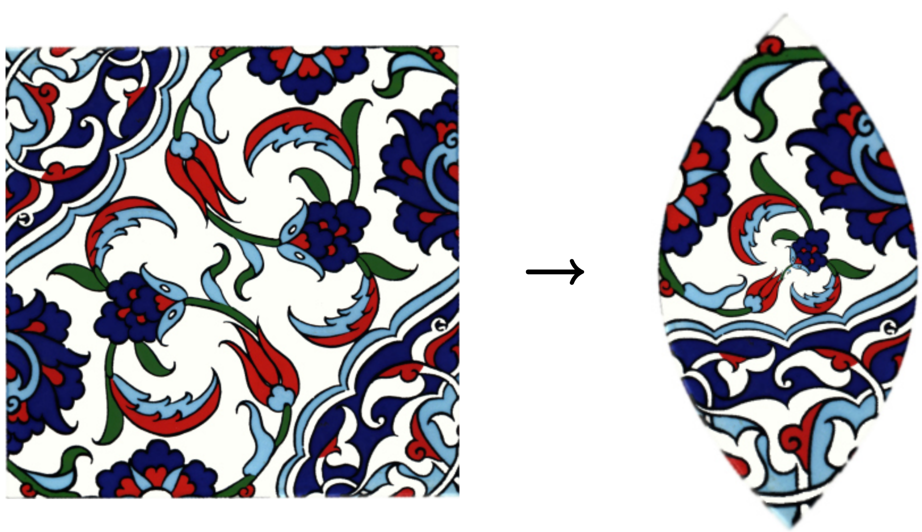

Figure 5: *Push-forward of a ceramic tile through the map* $z \mapsto z^2$.

An example of this push-forward is given in Figure 5. The ceramic tile on the left has a point reflection symmetry at the center, so we can push it through the map that sends z to z^2, and obtain the picture on the right. More precisely, to color a point z on the right, one looks at the two corresponding points on the left that map to it, which we could call $\sqrt{z}$ and $-\sqrt{z}$. The colors at the opposite points $\sqrt{z}$ and $-\sqrt{z}$ are the same, and this is the color we give to the point z on the right. Opposite corners of the tile have the same image, so the push-forward has only two corners. Since the transformation is conformal and the angles in the tile are 90 degrees, the push-forward is a 2-gon with 90 degree angles.

5. The exponential transform

Figure 6: *Exponential transform of a tea pot.*

Given the formula for squaring complex numbers, the transformation in Figure 4 is easy to apply. However, we saw that without knowledge of complex numbers one can arrive at the same result if one starts with the idea of wrapping a circle around itself twice and extends that process to a fine grid of small squares in a conformal way.

Similarly, we can take a line in the plane, say a vertical line, and wrap it around a circle. This transformation extends conformally to the entire plane too, and this way we obtain the *complex exponential map* that sends z to $\exp(z)$. By the *exponential transform* of a picture on a plane with a marked point we will mean the pull-back over the complex exponential map. This transformation occurs in Escher's work; see for instance his "Path of Life" series (1958), together with his periodic drawings number 102 and 125 [4, p. 311, 316], and his colored circular design with butterflies, together with periodic drawing number 79 [4, p. 305], [5, p. 76, 77]. Escher may well have found this transformation by the process of conformal continuation.

Another way to understand the complex exponential map is as a suitable limit of maps of nth powering of complex numbers. To see this in terms of transformations of pictures, let us consider the tea pot in the previous section. Figure 4 shows the pull-back for the conformal map that wraps a circle around itself twice. We could also wrap the circle 3 times, or more times, say n times, around itself. This map sends z to z^n. The pull-back over this map shows a circular chain of n tea pots, which are linked by n handles. If we want to view the tea pots in roughly the same size, then we need to enlarge the circle in order to fit n tea pots in. As n gets very large, this circle gets very big. We can then look on the pull-back where the tea pots will look as if they are aligned in a vertical line. In the limit they do lie along a straight line, see Figure 6. This process, which is well suited for animation, illustrates the mathematical fact

$$\exp(z) = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n.$$

If we rotate the original tea pot in a counterclockwise direction around the chosen center inside the handle, then the exponential transform will just get translated vertically upwards. Thus, the exponential transform of any picture has a symmetry property: it is *periodic*. By this we mean that there is a translation, by a so-called *period* of the picture, that leaves the picture invariant. For the exponential transform of a picture the period in the vertical direction corresponds to rotating the original picture full circle. In the usual complex coordinate this is a vertical shift over distance 2π.

Perhaps more strikingly, scaling the original picture has the effect of a horizontal translation on the exponential transform. One can see this either with formulas for the exponential map, or with our defining property of being the unique conformal extension of the map wrapping a line around a circle. This brings us to the main point: *the exponential transform of a Droste picture is a doubly periodic picture.* By a doubly periodic picture we mean a picture that has translation symmetries in distinct directions, such as pictures used on most wallpapers and tessellations. In fact, when we normalize distances in the usual way, so that rotating the Droste picture by 360 degrees is the vertical shift by 2π on the exponential transform, then the horizontal period of the exponential transform is exactly the natural logarithm of the scaling factor of the Droste picture.

Figure 7: *Exponential transform of the Droste picture in Figure 1.*

Conversely, suppose that we have a periodic picture with a given period, i.e., a picture on the plane which is invariant under a given translation. Let us choose a point P and draw a line L through P in the direction of the translation. We can then wrap L around a circle in such a way that the piece of L connecting P to its translate covers the circle exactly once in a counterclockwise direction. We make the conformal continuation of this map to the entire plane and construct the push-forward of the picture. We obtain a picture on the plane with a marked point: the center of the circle. One can show that a different choice of P will only change the push-forward by scaling and rotating with the marked point as the center.

One way to visualize this push-forward is by thinking of a pattern on a carpet which is preserved under a given translation. Roll up the carpet in such a way that each point falls exactly on its translate. Now hold up this cylinder and view the pattern on the inside of the cylinder. Straight lines on the carpet in the direction of the translation have become concentric circles. This process does not really produce a conformal transformation, but it does illustrate some aspects of the transformation. In the picture below we take a vertical period of three steps.

Figure 8: *Push-forward of a doubly periodic picture through the exponential map.*

6. Twisted Droste pictures

We have seen that pulling back over the exponential function transforms a Droste picture into a doubly periodic picture. To make a push-forward of this doubly periodic picture, we have to choose a period, i.e., a direction for rolling up the carpet. If we take the standard vertical period of 2π, which corresponds to rotating the Droste picture full circle, then the push-forward we obtain is again the same Droste picture. But we can also choose other periods, for instance a horizontal period that corresponds to zooming in by the Droste-factor on the Droste picture. For Escher's *Print Gallery* the chosen period corresponds to rotating full circle and also zooming out by the scaling factor. For our cruise ship this gives rise to the picture in Figure 9.

Figure 9: *Cruise ship with pool – twisted.*

Pushing forward a double periodic picture by "rolling up" over a certain period removes the symmetry of that period, but we still have another period in an independent direction. This other period gives rise to a symmetry on the push-forward: it is a symmetry which both rotates and scales the image. In Figure 9 it is a rotation by 154 degrees and scaling by a factor of about 22. This may be viewed as a twisted Droste-effect: this is the effect we would have when the nurse in Figure 1 puts the box of cocoa on her tray on its side.

There are many choices of the period, and they all give rise to different *twists* of our Droste picture. See [1] for many variants of Escher's lithograph obtained this way. Next we will see how to interpolate between different twists.

7. Dehn twists

As we have seen, pushing a picture forward through a transformation often removes symmetries. For the doubly periodic pictures, we can remove the periodicity by folding the picture onto itself, so that each point is identified with all its translates. The resulting space, which we still need to view with conformal structure, i.e., a local notion of angles at each point, is called a Riemann surface of genus 1. It looks like the surface of a donut, or an inner tube. One can think of such a space by starting with a parallelogram identifying each side with its opposite side by translation. The periods of the doubly periodic picture can be thought of as closed paths of rubber bands on the surface of the donut. Moving a rubber band without taking any part off the donut surface does not change the period.

One can also view the donut as the space that one obtains from the Droste picture by first drawing a circle around the center. Then one erases everything inside the circle, and everything outside the circle blown up by the scaling factor of the Droste picture. A ring shaped picture remains, and one gets the donut by lifting the inner and outer border out of the plane, and sowing them together. Thus, we also obtain a map from the Droste picture to the donut.

If we cut the inner tube so that it becomes a cylinder, then we can fix it by gluing two circles together again. The *Dehn twist* can be thought of as a gradual process where we glue the two circles only after rotating one of them, first by a zero angle, so nothing happens, but then by a gradually increasing angle, until we have come full circle. After the Dehn twist, what has changed? If we view the tube without its ambient three-dimensional space, then after the Dehn twist it is exactly the same tube again: the two circles are identified in the same way as before. However, if we also keep track of a closed path on the tube, then the path has changed.

Thus, the Dehn twist gives a continuous way to change a picture on a donut, together with a path on it, which at the end of the process leaves us with the same picture but a different path. The price we have to pay is that we need to cut the donut: this cut is visible in Figure 10. If at each stage of the Dehn twist on the donut, we pull back to the plane of doubly periodic pictures, and then push forward through the exponential map, then we can obtain a continuous animation that interpolates between different twists of the Droste picture. It turns out that any two twists of the same Droste picture can be connected by a sequence of such Dehn twists, with varying cuts in the donut surface.

Figure 10: *Dehn twist.*

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