

# Countable state Markov processes: non-explosiveness and moment function

F.M. Spieksma

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## Abstract

The existence of a moment function satisfying a drift function condition is well-known to guarantee non-explosiveness of the associated minimal Markov process (cf.[1, 6]), under standard technical conditions. Surprisingly, the reverse is true as well for a countable space Markov process. We prove this result by showing that recurrence of an associated jump process, that we call the  $\alpha$ -jump process, is equivalent to non-explosiveness. Non-explosiveness corresponds in a natural way to the validity of the Kolmogorov integral relation for the function identically equal to 1. In particular, we show that the  $\alpha$ -jump chain is positive recurrent, all bounded functions satisfy the Kolmogorov integral relation. Positive recurrence can be characterised by a drift function criterion as well.

If to a drift function  $V$ , there corresponds another drift function  $W$ , which is a moment with respect to  $V$ , then via a transformation argument, the above relations hold for the transformed process with respect to  $V$ . Transferring the results back to the original process, allows to characterise which  $V$ -bounded functions satisfy the Kolmogorov forward equation.

KEYWORDS Explosiveness of Markov Processes, drift conditions, moment function,  $\alpha$ -jump chain,  $V$ -transformation.

## 1 Introduction

It is a truth universally acknowledged that truths leaping to the eye are obscured by their own proximity. We were surprised to realise that the existence of a moment function for a minimal Markov process satisfying a drift condition is not only sufficient but also necessary for non-explosiveness.

The proof is actually quite simple and relies on a characterisation of non-explosiveness of a Markov process by recurrence of so-called  $\alpha$ -jump chains. Let us make these statements more precise as well as their motivation.

Consider a continuous time Markov process with values in a countable state space  $\mathbf{S}$ . We will restrict our analysis to Markov processes satisfying the following assumption.

**Assumption 1.1**  $X$  is a minimal, standard, stable Markov process with right-continuous sample paths (with respect to the discrete topology), and with conservative  $q$ -matrix, in other words  $Q = (q_{xy})_{x,y \in \mathbf{S}}$ , such that for all  $x \in \mathbf{S}$

1.  $0 \leq q_x = -q_{x,x} < \infty$ ;
2.  $\sum_y q_{xy} = 0$ .

The assumptions imply (cf. [1] Theorem 2.2.2) that the associated transition functions  $\{P_t = (p_{t,xy})_{x,y \in \mathbf{S}}\}_t$  satisfy the Kolmogorov forward and backward equations, and that they are the minimal non-negative solutions of these equations. Following the terminology in [6] we use the term *non-explosiveness* instead of the more commonly used *regularity* to denote that the process does not explode in finite time.

In [12] we have studied conditions on functions  $f : \mathbf{S} \rightarrow \mathbf{R}$ , in order to satisfy an integral form of the Kolmogorov forward equations. For non-negative functions  $f$  that satisfy the drift inequality  $Qf \leq cf$  (componentwise) for some constant  $c \in \mathbf{R}$ , it has been proved that the Kolmogorov forward integral equation

$$P_t f = f + \int_0^t P_s(Qf) ds \tag{1.1}$$

holds if and only if an associated Markov process  $X^f$ , obtained from  $X$  by a transformation induced by  $f$  (see Section 2) is non-explosive. This is even true when  $X$  itself is explosive. This is of importance when considering functions of the Markov process, say reward functions associated with a queueing process. It is therefore of interest to have direct tools for characterising non-explosiveness. To my knowledge the easiest applicable tool is the one described below. We use the notation  $\mathbf{R}_+$  for the non-negative reals i.e.  $\mathbf{R}_+ = [0, \infty)$ .

**Definition 1.1** •  $V : \mathcal{S} \rightarrow \mathbf{R}_+$  is called a *moment function*, if there exists an increasing sequence  $\{K_n\}_n$  of finite sets, i.e.  $K_n \subset \mathcal{S}$ ,  $K_n \subset K_{n+1}$ ,  $n = 1, 2, \dots$ , such that  $\inf_{x \notin K_n} V(x) \rightarrow \infty$ ,  $n \rightarrow \infty$ .

- $V : \mathcal{S} \rightarrow \mathbf{R}_+$  is called a *c-drift function*, if  $QV \leq cV$ , where “ $\leq$ ” stand for component wise ordering and  $c \in \mathbf{R}$ .

Note that in [6] a moment function is called a norm-like function. It has been proved for countable state Markov processes in [1], and for general state space processes in [6] that the existence of a  $c$ -drift moment function, for some constant  $c \in \mathbf{R}$ , implies non-explosiveness of the Markov process. Surprisingly, the reverse is also true. This will be the main theorem of our paper, formulated below. Before doing so, we will give the definition of explosiveness.

To this end, let  $\tau_0 = 0$ , and  $\tau_{n+1} = \inf\{t > \tau_n \mid X_t \neq X_{t-}\}$ , if  $X_{\tau_n}$  is non-absorbing. If  $X_{\tau_n}$  is absorbing, then put  $\tau_k = \infty$  and  $X_{\tau_k} = X_{\tau_n}$ , for  $k > n$ . Let  $J_\infty = \lim_{n \rightarrow \infty} \tau_n$ . Then  $X$  is said to be *explosive*, if there exists  $x \in \mathcal{S}$  such that  $\mathbf{P}\{J_\infty < \infty \mid X_0 = x\} > 0$ .

**Theorem 1.2** *Let  $X$  be a minimal Markov process satisfying Assumption 1.1.  $X$  is non-explosive if and only if there exists a  $c$ -drift moment function  $V$  for  $X$  for some constant  $c \in \mathbf{R}$ .*

The result follows from a characterisation of non-explosiveness through recurrence of an associated discrete time Markov chain that we will develop shortly.

To this end, extend  $\mathcal{S}$  with a coffin state  $\delta$  and put  $\mathcal{S}_\delta = \mathcal{S} \cup \{\delta\}$ .

**Definition 1.3** Let  $p$  be a probability distribution on  $\mathcal{S}$ , with  $p_y > 0$  for all  $y \in \mathcal{S}_\delta$  and let  $\alpha > 0$ . The discrete time Markov chain  $X^\alpha$  is an  $\alpha$ -jump chain (with restart distribution  $p$ ) on the extended state space  $\mathcal{S}_\delta$ , if

1. the transition matrix, denoted  $P^\alpha$ , is given by

$$p_{xy}^\alpha = \begin{cases} \frac{q_{xy}}{q_x + \alpha}, & x, y, \in \mathcal{S}, x \neq y \\ \frac{\alpha}{q_x + \alpha}, & x \in \mathcal{S}, y = \delta \\ p_y, & x = \delta, y \in \mathcal{S}; \end{cases}$$

2. the restart distribution  $p$ , is such that  $\mathbf{E}_x \tau < \infty$  for all  $x \in \mathcal{S}$ , where  $\tau$  is the hitting time of state  $\delta$ , implies  $\sum_x p_x \mathbf{E}_x \tau < \infty$  as well as aperiodicity.

Note that  $X^\alpha$  is an irreducible, aperiodic Markov chain. We suppress the dependence on the restart distribution  $p$  in our notation, since it is irrelevant for the class properties transience, null recurrence and positive recurrence of the Markov chain, given the requirements imposed for  $p$ . For  $\alpha = 0$  an added coffin state is not necessary and we get the customary jump chain. The result is the following.

**Theorem 1.4** *Suppose that  $X$  satisfies Assumption 1.1.  $X$  is non-explosive if and only if for some  $\alpha > 0$  there exists a recurrent  $\alpha$ -jump chain. Under either condition, for each  $\alpha > 0$  there exists a recurrent  $\alpha$ -jump chain.*

A related result is a well-known assertion ([1], Proposition 5.3.2, [14]) that recurrence of the jump chain (0-jump chain) is equivalent to recurrence of the minimal Markov process. This ensures that recurrence of the jump chain implies the Markov process to be non-explosive. The reverse relation does not hold. As an example consider a birth-death process on the non-negative integers, with birth rates  $\lambda_i = 2/3$  for  $i = 0, \dots$  and death rates  $\mu_i = 1/3$ ,  $i > 0$ . This has a transient jump chain. However, the rates are bounded and hence the Markov process is non-explosive.

The characterisation of non-explosiveness via the existence of a  $c$ -drift moment function allows to shed some light on various drift criteria for non-explosiveness from the literature. In section 4 we will compare this

result with drift criteria from [8, 3]. In the paper [12] we have discussed how non-explosiveness properties are related to the validity of the Kolmogorov forward integral relation. In view of the above theorems the result from [12] can be sharpened. In Section 5 we will discuss this and compare the result to a criterion used in [4, 10, 9].

We use the notation  $\mathbf{R}_+ = [0, \infty)$  and  $\mathbf{R}_{>0} = (0, \infty)$ .

## 2 Proof of the main theorems

We will first recall some relevant properties of a Markov process satisfying Assumption 1.1. In the first place, [1] Proposition 1.2.4 shows that  $t \mapsto p_{t,xy}$  is continuously differentiable on  $[0, \infty)$ , and  $x, y \in \mathbf{S}$ . Moreover,  $t \mapsto \sum_y p_{t,xy}$  is non-increasing and by virtue of [1] Proposition 1.2.6 it is continuously differentiable on  $[0, \infty)$  for each  $x \in \mathbf{S}$ .

Since we are also interested in reward processes, the question arises whether  $t \mapsto P_t V(x) = \sum_y p_{t,xy} V(y)$  is continuous, provided of course that  $V : \mathbf{S} \rightarrow \mathbf{R}$  is integrable w.r.t. the kernel  $P_t$ ,  $t \geq 0$ . This is true under the assumption that  $V$  is a  $c$ -drift function, for some constant  $c \in \mathbf{R}$ . Indeed, by virtue of [1] Proposition 2.2.13

$$P_t V \leq e^{ct} V, \quad t \geq 0, \quad (2.1)$$

since  $X$  is the minimal process, if  $V$  is a  $c$ -drift function. We now invoke a construction from [12], see also [1] Lemma 5.4.2. With  $X$  we associate the minimal Markov process  $X^V$  with  $q$ -matrix  $Q^V$ .  $X^V$  is called a  $V$ -transformation of  $X$  (it is not unique, since for each larger constant than  $c$  we can define such transformation as well). Again, we work with the extended state space  $\mathbf{S}_\delta$ . Then define

$$q_{xy}^V = \begin{cases} q_{xy} V(y)/V(x), & x \neq y; x, y \neq \delta \\ q_{xx} - c, & x = y; x, y \neq \delta \\ c - \sum_{y \in \mathbf{S}} q_{xy} V(y)/V(x), & y = \delta; x \neq \delta \\ 0, & y \in \mathbf{S}_\delta; x = \delta \end{cases}$$

with  $\delta_{xy}$  the Kronecker delta. This makes  $Q^V$  a conservative  $q$ -matrix, with  $\delta$  an absorbing state. Denote by  $\{P_t^V\}_t$  again the (minimum) transition function on the enlarged state space  $\mathbf{S} \cup \{\delta\}$ .

As has been pointed out above,  $t \mapsto p_{t,xy}^V$ ,  $t \mapsto \sum_{y \in \mathbf{S}_\delta} p_{t,xy}^V$  are continuously differentiable on  $[0, \infty)$ , for  $x, y \in \mathbf{S}_\delta$ . Note that ([1], Lemma 5.4.2)

$$p_{t,xy}^V = e^{-ct} p_{t,xy} \frac{V(y)}{V(x)}, \quad x, y \in \mathbf{S}, \quad (2.2)$$

which actually sharpens (2.1). As a consequence of this transformation we can transfer results for a Markov process to its expectation with respect to  $V$ . It then follows that  $t \mapsto P_t V(x)$  is continuously differentiable on  $[0, \infty)$ , for each  $x \in \mathbf{S}$ . As a side-remark, in [12] I wrote that continuity of  $t \mapsto P_t V(x)$  follows from (2.1) by virtue of the dominated convergence theorem. This, unfortunately, seems to imply only right-continuity.

By the dominated convergence theorem  $t \mapsto P_t f$  is continuous for any function  $f : \mathbf{S} \rightarrow \mathbf{R}$  with  $\sup_x |f(x)|/V(x) < \infty$ . This fact calls for resorting to Banach space terminology here. Introduce the norm  $\|\cdot\|_V$  of functions  $f : \mathbf{S} \rightarrow \mathbf{R}$  by

$$\|f\|_V = \sup_{x \in \mathbf{S}} \frac{|f(x)|}{V(x)}.$$

Then

$$\ell^\infty(\mathbf{S}, V) = \{f : \mathbf{S} \rightarrow \mathbf{R} \mid \|f\|_V < \infty\}$$

is a Banach space. The validity of (2.1) under Assumption 1.1 for a  $c$ -drift function  $V$  implies that  $P_t : \ell^\infty(\mathbf{S}, V) \rightarrow \ell^\infty(\mathbf{S}, V)$  is a bounded linear operator with respect to this norm.

The above discussion now leads to the following conclusion: *if  $V$  is a  $c$ -drift function, then  $t \mapsto P_t f$  is componentwise continuously differentiable for each  $f \in \ell^\infty(\mathbf{S}, V)$ .*

Before turning to prove Theorems 1.2 and 1.4, we need introduce the resolvent. Let  $\alpha > 0$ . The resolvent  $R(\alpha)$ ,  $\alpha > 0$  is defined by

$$R_{xy}(\alpha) = \int_0^\infty e^{-\alpha t} p_{t,xy} dt,$$

which can be interpreted as the expected time spent in state  $y$  before “extinction or explosion”, when the initial state is  $x$ . Extinction in this case is interpreted to occur at rate  $\alpha$ . One can associate a non-conservative  $q$ -matrix with the resolvent, such that the corresponding jump chain is the restriction of any  $\alpha$ -jump chain to  $\mathbf{S}$ .

Next we write  ${}_{\delta}P^{\alpha}$  for the taboo matrix, with taboo state  $\delta$ , ignoring the jumps towards  $\delta$ . That is,  ${}_{\delta}p_{xy}^{\alpha} = 0$ ,  $y = \delta$ , and  ${}_{\delta}p_{xy}^{\alpha} = p_{xy}^{\alpha}$  for  $y \in \mathbf{S}$ . Put  ${}_{\delta}P^{\alpha, (0)} = \mathbf{I}$ , where  $\mathbf{I}$  is the identity matrix, and generally  ${}_{\delta}P^{\alpha, (t)}$  for the  $t$ -th iterate. The taboo transition probabilities for states in  $\mathbf{S}$  are the transition probabilities of the “non-extended” (i.e. without  $\{\delta\}$ )  $\alpha$ -jump chain. We will first prove Theorem 1.4.

*Proof of Theorem 1.4.* From [1] Proposition 4.1.1

$$R_{xy}(\alpha) = \sum_{t=0}^{\infty} {}_{\delta}p_{xy}^{\alpha, (t)} \frac{1}{\alpha + q_y}. \quad (2.3)$$

By virtue of [1], Theorem 2.2.2, Propositions 2.2.3, 2.2.4,  $X$  is explosive if and only if  $\sum_y p_{t,xy} < 1$  for some state  $x \in \mathbf{S}$  and some  $t \geq 0$ . Since this is a non-increasing function of  $t$ , the following fact is simply derived (cf. [1] Proposition 1.3.1): *given  $\alpha > 0$ , the matrix  $\alpha R(\alpha)$  is a stochastic matrix if and only if  $P_t$  is a stochastic matrix for some  $t > 0$ . In either case  $\alpha R(\alpha)$  is stochastic for all  $\alpha > 0$  and  $P_t$  is stochastic for all  $t \geq 0$ .*

Multiplying (2.3) by  $\alpha$ , it follows that

$$\alpha \sum_{y \in \mathbf{S}} R_{xy}(\alpha) = \sum_{t=0}^{\infty} {}_{\delta}p_{\alpha, xy}^{(t)} \frac{\alpha}{\alpha + q_y} = \mathbb{P}\{\exists n \geq 1 \text{ s.t. } X_n^{\alpha} = \delta \mid X_0^{\alpha} = x\},$$

is the probability of reaching  $\delta$  in the extended  $\alpha$ -jump chain.

This means that the row sums of  $\alpha R(\alpha)$  equal 1 if and only if state  $\delta$  is reached with probability 1 from every state  $x \in \mathbf{S}$ , in the Markov chain  $X^{\alpha}$ . By irreducibility, this is true if and only if  $X^{\alpha}$  is a recurrent Markov chain. QED

By virtue of the above result, the proof of Theorem 1.2 now reduces to applying the necessary and sufficient condition for recurrence of irreducible, aperiodic discrete time Markov chains given in [5] Theorem 1, the ‘if’ part of which is very well-known. However, the ‘only if’ part of this result does not seem to be as widely known and so we will quote the precise result here. We will also give the proof of the ‘only if’ part here, first because the formulation is slightly different and secondly because it has some interesting consequences.

**Theorem 2.1 ([5] Theorem 1)** *Let  $X = \{X_n\}_{n=0, \dots}$  be a discrete time irreducible, aperiodic Markov chain on a countable state space  $\mathbf{S}$  with transition matrix  $P$ .  $X$  is recurrent if and only if there exists a moment function  $f : \mathbf{S} \rightarrow \mathbf{R}_+$  and a finite set  $K \subset \mathbf{S}$ , such that*

$$Pf(x) \leq f(x), \quad x \notin K. \quad (2.4)$$

*In particular, under either condition the following properties hold:*

- i) *for any finite set  $K \subset \mathbf{S}$  there exists a moment function  $f : \mathbf{S} \rightarrow \mathbf{R}_+$  satisfying (2.4) with  $f(x) = 0$ ,  $x \in K$ ;*
- ii) *for any increasing sequence of finite sets  $\{K_n\}_n$ ,  $K_n \uparrow \mathbf{S}$ , there exists a function  $f : \mathbf{S} \rightarrow \mathbf{R}_+$ , with  $\inf_{x \notin K_n} f(x) \rightarrow \infty$ , as  $n \rightarrow \infty$ , and  $f(x) = 0$ ,  $x \in K_1$ , such that (2.4) is satisfied for  $f$  and the set  $K_1$ .*

*Proof of the ‘only if’ part.* We assume that  $X$  is recurrent. Let  $K \subset \mathbf{S}$  be an arbitrary finite set. Let  $K_n \supset K$ ,  $n = 1, \dots$ , be any increasing sequence of finite sets, with  $K_n \uparrow \mathbf{S}$  as  $n \rightarrow \infty$ . Define  $\tau_n = \inf\{t \geq 0 \mid X_t \in K_n^C\}$  and  $\tau_K = \inf\{t \geq 0 \mid X_t \in K\}$ .

Define

$$\phi^n(x) = \mathbb{P}\{\tau_n \leq \tau_K \mid X_0 = x\}.$$

Then  $\phi^n(x) = 0$  for all  $n$ , and  $x \in K$ . On the other hand,  $\phi^n(x) = 1$  for  $x \in K_n^C$ . Consequently

$$P\phi^n(x) = \sum_y p_{xy} \phi^n(y) \leq \phi^n(x), \quad x \notin K. \quad (2.5)$$

Hence  $\phi^n$  satisfies (2.4) for the set  $K$ . However,  $\phi^n$  is bounded. We will construct a moment function by summing an appropriate collection of the functions  $\phi^n$ . To this end, let  $x \notin K$  be given. Clearly  $n \mapsto \phi^n(x)$  is a non-increasing function of  $n$ , since  $\{\tau_n \leq \tau_K\} \supset \{\tau_{n+1} \leq \tau_K\}$ .

Then  $\{\tau_n \leq \tau_K\} \supset \{X_t \notin K, t \geq 0\}$ , since the Markov chain restricted to states of  $K_n \setminus K$  is transient. Consequently,  $\bigcap_n \{\tau_n \leq \tau_K\} \supset \{X_t \notin K, t \geq 0\}$ . We claim that

$$\bigcap_n \{\tau_n \leq \tau_K\} = \{X_t \notin K, t \geq 0\}.$$

It is sufficient to prove that

$$\bigcup_n \{\tau_n > \tau_K\} \supset \{\exists t \geq 0 \text{ such that } X_t \in K\}.$$

This is clearly true, because each trajectory leading to set  $K$  has finite ‘length’. Now, by monotone convergence and recurrence, we have that

$$\lim_{n \rightarrow \infty} \mathbb{P}\{\tau_n \leq \tau_K \mid X_0 = x\} = 0, \text{ for each } x \in \mathbf{S}.$$

Let  $n_0 = 0$ . For any  $k \geq 1$  choose index  $n_k > n_{k-1}$  so that  $\phi^{n_k}(x) \leq 2^{-k}$  for  $x \in K_k$ . Take  $n_0 = 0$ . Define

$$f(x) = \sum_{k \geq 1} \phi^{n_k}(x).$$

$f(x)$  is finite for each  $x \notin K$ , and  $f(x) = 0$  for  $x \in K$ . Moreover  $f(x) \geq k$  for  $x \in K_{n_k}^C$ . Hence  $f$  is a moment function. Further,  $f$  satisfies (2.4) since each term in the summation satisfies (2.4) by virtue of (2.5).

The assertions (i) and (ii) follow from the above construction. QED

*Proof of Theorem 1.2.* By virtue of Theorem 1.4 it is sufficient to show that the  $\alpha$ -jump chain  $X^\alpha$  is recurrent if and only if there exists a  $\alpha$ -drift moment function for  $X$ .

Suppose that such a function exists. Denote it by  $V$ . Then  $QV \leq \alpha V$  implies that  ${}_\delta P^\alpha V(x) \leq V(x)$  for  $x \neq \delta$ . Setting  $V(\delta) = 0$ , we get  ${}_\delta P^\alpha V(x) \leq V(x)$ ,  $x \neq \delta$ . Hence (2.4) is satisfied for this function  $V$  and set  $K = \{\delta\}$ , and so  $X^\alpha$  is recurrent by virtue of Theorem 2.1.

Vice versa, suppose that  $X^\alpha$  is recurrent. By Theorem 2.1(i) it follows that there exists a moment function  $V$  with  $V(\delta) = 0$ , such that  ${}_\delta P^\alpha V(x) \leq V(x)$  for  $x \neq \delta$ . Hence  $QV(x) \leq \alpha V(x)$ ,  $x \in \mathbf{S}$ . QED

The next sections discuss consequences of the main theorems and relations with drift criteria concerning non-explosiveness and the Dynkin formula from the literature.

### 3 Existence of bounded eigenvectors for $Q$

The paper [11] Theorem 7 states that a conservative Markov process is non-explosive if and only if for no value  $\alpha > 0$  the  $q$ -matrix has a non-zero bounded eigenvector to eigenvalue  $\alpha$ . We give an alternative proof.

**Lemma 3.1** *Suppose that  $X$  satisfies Assumption 1.1.  $X$  is non-explosive if and only if for no value  $\alpha > 0$  the  $q$ -matrix has a bounded non-zero (right) eigenvector to eigenvalue  $\alpha$ .*

*Proof.* Suppose that  $X$  is non-explosive and suppose that  $g : \mathbf{S} \rightarrow [-1, 1]$ ,  $g \not\equiv 0$ , is an eigenvector of  $Q$  to eigenvalue  $\alpha > 0$ . In other words  $\alpha g = Qg$ . Then

$$g(x) = \sum_{y \in \mathbf{S}} p_{xy}^\alpha g(y) = \sum_y {}_\delta p_{xy}^\alpha g(y), \quad x \neq \delta,$$

where we take  $g(\delta) = 0$ . By iteration

$$g(x) = \sum_y {}_\delta p_{xy}^{\alpha, (n)} g(y), \quad n \geq 1, x \in \mathbf{S}.$$

Let  $\tau = \min\{n \mid X_n^\alpha = \delta\}$  be the hitting time of  $\delta$ . For  $x \neq \delta$

$$-\mathbb{P}\{\tau > n \mid X^\alpha = x\} = -\sum_y {}_\delta p_{xy}^{\alpha, (n)} \leq g(x) \leq \sum_y {}_\delta p_{xy}^{\alpha, (n)} \leq \mathbb{P}\{\tau > n \mid X^\alpha = x\}, \quad n \geq 1, x \in \mathbf{S}. \quad (3.1)$$

If  $X$  is non-explosive, the  $\alpha$ -jump chain is recurrent, and hence both right-hand and left-hand sides of (3.1) converge to 0, as  $n \rightarrow \infty$ . Hence for  $Q$  no bounded eigenvector to the positive eigenvalue  $\alpha$  can exist. On the other hand, suppose  $Q$  has a bounded eigenvector  $g \neq 0$  to eigenvalue  $\alpha > 0$ . We may assume that  $\sup_x |g(x)| \leq 1$ . Then there exists  $x \in \mathcal{S}$  such that  $g(x) \neq 0$ , say  $g(x) > 0$ . It follows that  $\mathbb{P}\{\tau > n \mid X^\alpha = x\} \geq g(x) > 0$  for all  $n$ . Hence the  $\alpha$ -jump chain is transient, and so  $X$  is explosive. QED

**Corollary 3.2** *Under the conditions of Lemma 3.1, either  $Q$  has a bounded eigenvector to eigenvalue  $\alpha$  for each  $\alpha > 0$ , or for none.*

Suppose that  $V$  is a  $c$ -drift function. In [12] it was shown that  $g \in \ell^\infty(\mathcal{S}, V)$  is an eigenvector of  $Q$  to eigenvalue  $\alpha > 0$  if and only if  $f : \mathcal{S}_\delta \rightarrow \mathbf{R}$  defined by  $f(x) = g(x)/V(x)$ ,  $x \in \mathcal{S}$ ,  $f(\delta) = 0$ , is a bounded eigenvector of the  $V$ -transformation  $Q^V$ , for  $\alpha > 0$ . Since  $\delta$  is an absorbing state for the  $V$ -transformation, any eigenvector to  $Q^V$  must have value 0 at  $\delta$ .

Hence,  $X^V$  non-explosive is equivalent to the non-existence of eigenvectors of  $Q$  to any eigenvalue  $\alpha > 0$ , in the space  $\ell^\infty(\mathcal{S}, V)$ .

## 4 Drift criteria

Recently a new criterion for non-explosiveness has been proposed, for a Markov process with a general state space [8]. The assertion recast in our notation reads as follows.

**Lemma 4.1 ([8] Corollary 3.2)** *Suppose that  $Q$  is conservative and satisfies Assumption 1.1. Let  $\alpha > 0$ . Assume the existence of a non-negative function  $V : \mathcal{S} \rightarrow \mathbf{R}_+$  such that*

$$\sum_{y \in \mathcal{S}} q_{xy} V(y) + q_x \leq \alpha V(x), \quad x \in \mathcal{S}. \quad (4.1)$$

*Then  $X$  is non-explosive. In this case the minimal non-negative solution of (4.1) is given by*

$$V_{\min}(x) = \sum_{t=0}^{\infty} \sum_y \delta p_{xy}^{\alpha, (t)} \frac{q_y}{\alpha + q_y} = \sum_y R_{xy}(\alpha) q_y. \quad (4.2)$$

This lemma does not require that  $V$  be a moment function. By virtue of Theorem 1.2 necessarily the condition in the lemma implies the existence of a  $c$ -drift moment function. The question of interest is whether (4.1) is stronger than non-explosiveness or equivalent to it.

Using  $\tau$  again to denote the first hitting time of state  $\delta$ ,

$$V_{\min}(x) = \sum_{t=0}^{\infty} \sum_y \delta p_{xy}^{\alpha, (t)} \frac{q_y}{\alpha + q_y} = \sum_{t=0}^{\infty} \mathbb{P}\{\tau > t \mid X^\alpha(0) = x\} = \mathbf{E}\{\tau \mid X^\alpha(0) = x\}, \quad x \neq \delta.$$

Hence, if there is a non-negative function satisfying (4.1) then there  $\alpha$ -jump chain is positive recurrent. Vice versa, if the  $\alpha$ -jump chain is positive recurrent, then  $V(x) = \mathbf{E}\{\tau \mid X^\alpha(0) = x\}$  is finite and hence a solution to (4.1). Notice that  $V_{\min} \geq 1$  and so any solution to (4.1) has the property that it is bounded below by 1!

**Corollary 4.2** *The existence of a non-negative function  $V : \mathcal{S} \rightarrow \mathbf{R}_+$  satisfying (4.1) is equivalent to positive recurrence of the corresponding  $\alpha$ -jump chain. For any such solution  $V$  it holds that  $V \geq 1$ .*

Below we will give two examples. One where the  $\alpha$ -jump chains are all recurrent, they are null-recurrent for some values of  $\alpha$  and positive recurrent for others. In the second example all  $\alpha$ -jump chains are null-recurrent. This illustrates that the condition in Lemma 4.1 is not equivalent to non-explosiveness.

**Example 4.3** Let  $\mathcal{S} = \{1, \dots\}$ . We consider the minimal Markov process  $X$  with  $q$ -matrix with the only non-zero entries given by  $q_{x, x+1} = x = -q_{x, x}$ ,  $x \in \mathcal{S}$ .  $X$  is a non-explosive pure birth process.

The latter follows from the fact that  $\sum_x q_x^{-1} = \infty$  (cf. [7], Theorem 2.5.2). Alternatively, choosing moment function  $V(x) = x$ , yields  $QV \leq V$ . Consequently,  $V$  is a 1-drift moment function.

Hence the  $\alpha$ -jump process is recurrent for any  $\alpha > 0$ . The expected time to reach the coffin state  $\delta$ , given the initial state is  $x$ , is equal to

$$\mathbb{E}\{\tau \mid X^\alpha(0) = x\} = \sum_{n=0}^{\infty} \prod_{t=0}^n \frac{x+t}{\alpha+x+t}.$$

Put  $\alpha = 1$ . Then the above reduces to

$$\sum_{n=0}^{\infty} \prod_{t=0}^n \frac{x+t}{1+x+t} = \sum_{n=0}^{\infty} \frac{x}{1+x+n},$$

which is divergent. As a consequence, the 1-jump chain is null-recurrent. For any  $\alpha \geq 2$  however it is simply deduced that the  $\alpha$ -jump chain is positive recurrent.

Note that the Markov process in the above example is a pure birth process, and so the associated jump chain (0-jump chain) is transient. However, the Markov process is non-explosive.

**Example 4.4** We refine the previous example slightly, leading to a non-explosive process for which no  $\alpha$ -jump chain is positive recurrent. Consider a pure birth process  $X$  on  $\mathbf{S} = \{1, 2, \dots\}$ , with rates  $q_{x,x+1} = x \ln x = -q_{xx}$ . Again we assume that  $X$  is the minimal process.

Similarly to the above example,  $\sum_x q_x^{-1} = \infty$ , and hence the process is non-explosive. A moment function does not seem easily constructed.

The expected time to reach the coffin state  $\delta$  by the  $\alpha$ -jump chain, given initial state  $x \in \mathbf{S}$  is equal to

$$\mathbb{E}\{\tau \mid X^\alpha(0) = x\} = \sum_{n=0}^{\infty} \prod_{t=0}^n \frac{(x+t) \ln(x+t)}{\alpha + (x+t) \ln(x+t)}. \quad (4.3)$$

Let  $\alpha > 0$  and  $x \in \mathbf{S}$  with  $\ln x \geq \alpha$ . Then  $(x+t) \ln(x+t) \geq (x+t)\alpha$  and hence

$$\frac{(x+t) \ln(x+t)}{\alpha + (x+t) \ln(x+t)} \geq \frac{x+t}{1+x+t}.$$

It follows that

$$\prod_{t=0}^n \frac{(x+t) \ln(x+t)}{\alpha + (x+t) \ln(x+t)} \geq \prod_{t=0}^n \frac{x+t}{1+x+t} = \frac{x}{x+n+1}, \quad n = 1, \dots,$$

so that (4.3) diverges. In other words,  $\mathbb{E}\{\tau \mid X^\alpha(0) = x\} = \infty$ , and so the  $\alpha$ -jump chain is null-recurrent for any  $\alpha > 0$ .

In [3] Assumption 2.2, Guo and Hernandez-Lerma formulate the following sufficient criterion for non-explosiveness: *there exist a function  $w : \mathbf{S} \rightarrow [1, \infty)$  and constants  $c_0 \neq 0$ ,  $b_0 \geq 0$  and  $L_0 > 0$  such that for  $x \in \mathbf{S}$*

$$\left. \begin{aligned} Qw(x) &\leq c_0 w(x) + b_0; \\ q_x &\leq L_0 w(x). \end{aligned} \right\} \quad (4.4)$$

The paper [8] (“*Discussion about condition 3.1*”) shows that this condition implies condition (4.1) for the function

$$V(x) = \frac{L_0}{\alpha - c_0} \left( w(x) + \frac{b_0}{\alpha} \right), \quad x \in \mathbf{S},$$

for any  $\alpha > c_0$ , which is easily checked. In particular it implies positive recurrence of the  $\alpha$ -jump chain for any  $\alpha > c_0$ . In view of Example 4.4 the condition is therefore sufficient but not necessary for non-explosiveness. It is easily verified that the requirement  $w(x) \geq 1$  is not used in this derivation.

The following example shows that this condition is stronger than (4.1).

**Example 4.5** Let  $X$  be a minimal Markov process on  $\{0, \dots\}$ , with  $q$ -matrix given by

$$q_{xy} = \begin{cases} p2^x, & y = x + 1 \\ (1-p)2^x, & y = x - 1, x \neq 0 \\ -2^x, & y = x > 0 \\ -p, & y = x = 0, \end{cases}$$

for some  $p$  with  $(3 - \sqrt{5})/2 < p < 1/2$ .

Let  $V : \mathbf{S} \rightarrow \mathbf{R}_+$  be given by  $V(x) = \beta^x$ , with  $\beta > 1$ . We claim that

- there exists  $\beta$  such that (4.1) is satisfied for the function  $V(x) = \beta^x$ ,  $x \geq 0$ ; and
- there is no function  $w$  satisfying (4.4).

Clearly,  $V$  is a moment function with  $QV \leq c \cdot V$ , with  $c = \max\{(p\beta + (1-p)/\beta - 1), p\beta - p\}$ . In particular, for  $x \geq 1$

$$QV(x) + q_x = (p\beta + (1-p)/\beta - 1)(2\beta)^x + 2^x = (2\beta)^x \left( p\beta + \frac{1-p}{\beta} - 1 + \beta^{-x} \right) \leq \alpha \beta^x,$$

for some constant  $\alpha$ , provided

$$p\beta + \frac{1-p}{\beta} - 1 + \beta^{-x} \leq 0,$$

for  $x$  large enough. This means that we should choose  $\beta > 1$ , such that

$$f(\beta) = p\beta + \frac{1-p}{\beta} - 1 < 0.$$

The zeroes of the polynomial  $p\beta^2 - \beta + 1 - p$  are 1 and  $(1-p)/p$ . This yields that

$$f(\beta) \begin{cases} < 0 & 1 < \beta < (1-p)/p, \\ > 0 & \beta > (1-p)/p. \end{cases}$$

The assumption  $(3 - \sqrt{5})/2 < p < 1/2$  implies that  $(1-p)/p \in (1, 2)$ . Hence, choosing  $\beta \in (1, (1-p)/p)$ , gives  $\beta < 2$  and so  $q_x > V(x)$ , for all  $x$ . This proves the first assertion.

We next prove the second assertion that there is no function  $w : \mathbf{S} \rightarrow \mathbf{R}_+$  satisfying (4.4). Suppose that not. In other words, there exists a function  $w : \mathbf{S} \rightarrow \mathbf{R}_+$ , with  $w(x) \geq 2^x$ , and  $Qw(x) \leq c'w(x) + d'$ . Since  $w$  is a moment function, this implies the existence of a constant  $c_0$  such that  $Qw \leq c_0w$ . We distinguish three cases.

**(a)** There exists an increasing sequence  $\{x_n\}_n$ , with  $x_1 \geq 1$ , and  $x_n \rightarrow \infty$ ,  $n \rightarrow \infty$ , such that  $w(x_n)/w(x_n - 1) \leq 2 \leq w(x_n + 1)/w(x_n)$ . Then

$$\frac{c_0}{2^{x_n}} \geq \frac{Qw(x_n)}{2^{x_n}w(x_n)} = p \frac{w(x_n + 1)}{w(x_n)} - 1 + (1-p) \frac{w(x_n - 1)}{w(x_n)} \geq 2p - 1 + \frac{1-p}{2} = f(2),$$

for all  $n$ . By the assumption  $f(2) > 0$ , so that the right-hand side of the above equation is strictly positive for all  $n$ . However, the left-hand side converges to 0, as  $n \rightarrow \infty$ . A contradiction.

**(b)**  $w(x + 1)/w(x) \geq 2$  for  $x \geq x_0$ , for some  $x_0 \in \mathbf{S}$ .

Put  $\theta_x = w(x + 1)/(2w(x))$ . Then  $\theta_x \geq 1$  for  $x \geq x_0$ . This implies

$$1 + \frac{c_0}{2^x} \geq 2p\theta_x + \frac{1-p}{2\theta_{x-1}}, \quad x \geq x_0. \quad (4.5)$$

For any  $\epsilon > 0$  there exists  $x(\epsilon) \geq x_0$  such that  $c_0/2^x \leq \epsilon$  for  $x \geq x(\epsilon)$ . We will later choose  $\epsilon$  appropriately.

By (4.5)  $\theta_{x-1} \leq (1 + \epsilon)/(2p)$  for  $x > x(\epsilon)$ . Combining  $\theta_x \geq 1$ , it follows that

$$(1 + \epsilon)^2 \geq 3p - p^2 + 2p\epsilon, \quad (4.6)$$

for  $x > x(\epsilon)$ . We have chosen  $p$  such that  $3p - p^2 > 1$ . Consequently, for  $\epsilon$  sufficiently small (4.6) is violated.

**(c)**  $w(x)/w(x - 1) \leq 2$  for  $x \geq x_0$ , for some  $x_0 \in \mathbf{S}$ .

Put  $\theta_x = 1 - w(x)/(2w(x - 1))$  for  $x \geq x_0$ . Then  $w(x)/w(x_0) = 2^{x-x_0} \prod_{y=x_0+1}^x (1 - \theta_y)$ . Since  $w(x) \geq 2^x$ , it follows that  $\prod_{y=x_0+1}^x (1 - \theta_y) \geq 2^{x_0}/w(x_0)$ ,  $x \geq x_0$ . Since  $\prod_{y=x_0+1}^x (1 - \theta_y)$  is non-increasing in  $x$ ,  $\prod_{y \geq x_0+1} (1 - \theta_y)$  must convergent as a function of  $x$ . In other words,  $\sum_{y \geq x_0+1} \theta_y$  is convergent (cf. [13] section 1.41). This implies that  $\theta_x \rightarrow 0$  as  $x \rightarrow \infty$ .

We now get for  $x > x_0$

$$\frac{c_0}{2^x} \geq \frac{Qw(x_n)}{2^x w(x_n)} = 2p(1 - \theta_x) - 1 + \frac{1-p}{2(1 - \theta_{x-1})}.$$

The left-hand side converges to 0, whereas the right-hand side converges to  $2p - 1 + (1-p)/2 = f(2) > 0$ , as  $x \rightarrow \infty$ . A contradiction.

## 5 Kolmogorov forward integral equation

In [12] non-explosiveness properties have been studied in connection to the problem for what functions  $f : \mathbf{S} \rightarrow \mathbf{R}$  the Kolmogorov forward integral equation holds:

$$P_t f(x) = f(x) + \int_0^t P_s(Qf)(x)ds = f(x) + \mathbf{E}_x \int_0^t Qf(X_s)ds, \quad x \in \mathbf{S}, \quad (5.1)$$

where  $P_t f(x) = \sum_y p_{t,xy} f(y)$ ,  $x \in \mathbf{S}$ . In case of  $f$  being a  $c$ -drift function for some constant  $c$ , this equation is holds if and only if the  $f$ -transformation  $X^f$  is non-explosive (cf. Theorem 3.2 [12]), provided Assumption 1.1 holds.

By Theorem 1.2  $X^f$  is non-explosive if and only there exists a  $d$ -drift moment function  $W$  for  $X^f$ , for some constant  $d$ . We may transform the function  $W$  to a function  $F : \mathbf{S} \rightarrow \mathbf{R}$ , given by  $F(x) = W(x)f(x)$ ,  $x \in \mathbf{S}$ , which will be called a  $f$ -moment function if  $W$  is a moment function for  $X^f$ .

**Definition 5.1** The function  $F : \mathbf{S} \rightarrow \mathbf{R}_+$  is called a  $f$ -moment function, if there exists an increasing sequence  $\{K_n\}_n$  of finite sets,  $K_n \uparrow \mathbf{S}$ , such that  $\inf_{x \notin K_n} F(x)/f(x) \rightarrow \infty$  as  $n \rightarrow \infty$ .

In view of Theorem 1.2 we may rephrase [12] Theorems 3.2 and 3.4.

**Corollary 5.2 to Theorem 1.2 and [12] Theorem 3.2.** *Suppose that Assumption 1.1 holds true and that  $V$  is a  $c$ -drift function for some constant  $c$ . Then  $V$  satisfies (5.1) if and only if there exists a  $d$ -drift  $V$ -moment function  $W$ , for some constant  $d$ .*

The following theorem specifies a class of functions that satisfy (5.1).

**Theorem 5.3 ([12] Theorem 3.4.)** *Let  $X$  satisfy Assumption 1.1. Suppose that  $V$  is a  $c$ -drift function for  $X$  and that  $W : \mathbf{S} \rightarrow \mathbf{R}_+$  is a  $d$ -drift  $V$ -moment function. Then  $V$  satisfies (5.1).*

*Additionally,  $f \in \ell^\infty(\mathbf{S}, V)$  satisfies (5.1) provided that  $\mathbf{E}_x \int_0^t |Qf(X_s)|ds < \infty$ ,  $x \in \mathbf{S}$ .*

Interestingly enough, posing the additional condition that the  $\alpha$ -jump chain  $X^{V,\alpha}$  is *positive recurrent* for some  $\alpha > 0$ , all functions  $f \in \ell^\infty(\mathbf{S}, V)$  satisfy the Kolmogorov integral equation (5.1). In the formulation of this result we use the drift condition characterisation of positive recurrence of  $\alpha$ -jump chains from the previous section applied to the  $V$ -transformation, but translate it to a drift condition for the original process  $X$ .

**Lemma 5.4** *Let  $X$  satisfy Assumption 1.1. Let  $V$  be a  $c$ -drift function for  $X$ . Then the following two conditions are equivalent.*

i) *There exists a function  $W : \mathbf{S} \rightarrow \mathbf{R}_+$  such that*

$$\sum_y q_{xy} W(y) + q_x V(x) \leq \alpha W(x), \quad x \in \mathbf{S}, \quad (5.2)$$

*for some  $\alpha > 0$ .*

ii) *The  $\alpha$ -jump chain  $X^{V,\alpha}$  associated with the  $V$ -transformation is positive recurrent for some  $\alpha > 0$ .*

*Proof.* First assume (i) the existence of a positive function  $W$  satisfying (5.2). Form the  $V$ -transformation and put  $W(\delta) = 0$ . Then  $X^V$  satisfies (4.1) for the function  $F : \mathbf{S} \cup \{\delta\} \rightarrow \mathbf{R}_+$  given by  $F(x) = W(x)/V(x)$ ,  $x \in \mathbf{S}$ , and  $F(\delta) = 0$  and constant  $\alpha - c$ . The equation for state  $\delta$  has 0 on both sides of the equation. Statement (ii) follows from Lemma 4.1.

Next assume (ii) that  $X^{V,\alpha}$  is positive recurrent. By the same Lemma there exists a non-negative function  $F$  for  $X^V$  satisfying (4.1). Then  $X$  satisfies (4.1) for the function  $W(x) = F(x)V(x)$  and constant  $\alpha + c$ . QED

**Theorem 5.5** *Let the assumptions and conditions of Lemma 5.4 be satisfied. Then  $\mathbf{E}_x \int_0^t |Qf(X_s)|ds < \infty$ ,  $x \in \mathbf{S}$ , for all  $f \in \ell^\infty(\mathbf{S}, V)$ , and so (5.1) applies for all  $f \in \ell^\infty(\mathbf{S}, V)$ .*

*Proof.* We will first consider the  $V$ -transformation  $X^V$  on the space  $\mathbf{S}_\Delta = \mathbf{S} \cup \{\Delta\}$ . By the previous Lemma and its proof  $X^{V,\alpha}$  is positive recurrent. Let us denote the exist state by  $\delta$ . By (4.2)

$$\sum_{y \in \mathbf{S}_\Delta} R_{xy}^V(\alpha) q_y^V = \sum_{y \in \mathbf{S}} R_{xy}^V(\alpha) (q_y + c) < \infty, \quad x \in \mathbf{S}.$$

Noting the definition of the resolvent  $R^V(\alpha)$  and using that  $c$  may be assumed positive, we get

$$\int_0^T \sum_{y \in \mathbf{S}} p_{t,xy}^V q_y dt \leq \exp^{\alpha T} \int_0^T \exp^{-\alpha t} \sum_{y \in \mathbf{S}} p_{t,xy}^V q_y dt < \infty, \quad x \in \mathbf{S}. \quad (5.3)$$

Since  $q_y + c \geq \sum_{z \in \mathbf{S}} q_{yz}^V$ , this implies

$$\int_0^T \sum_{y \in \mathbf{S}} \sum_{z \neq y, z \in \mathbf{S}} p_{t,xy}^V q_{yz}^V dt < \infty, \quad x \in \mathbf{S}. \quad (5.4)$$

Using (2.2), notice that for  $t \leq T$

$$\begin{aligned} p_{t,xy} V(y) q_y &\leq V(x) \exp^{cT} p_{t,xy}^V q_y \\ p_{t,xy} \sum_{z \neq y} q_{yz} V(z) &\leq V(x) \exp^{cT} p_{t,xy}^V \sum_{z \neq y, z \in \mathbf{S}} q_{yz}^V. \end{aligned}$$

Combining with (5.3) and (5.4) yields for  $x \in \mathbf{S}$

$$\int_0^T \sum_{y,z} p_{t,xy} \sum_z |q_{yz}| V(y) dt = \int_0^T P_t(|Q|V)(x) dt < \infty.$$

This implies for any function  $f \in \ell^\infty(\mathbf{S}, V)$  that

$$\int_0^T P_t |Qf|(x) \leq \int_0^T P_t (|Q|f)(x) dt < \infty,$$

for  $x \in \mathbf{S}$ . The results follows. QED

The problem of the validity of the Kolmogorov forward integral relation has been addressed in the context of the optimality equation of average optimality in Markov decision processes, see e.g. [4], the recent paper [10] and the book [9]. These works impose the following conditions for functions in order to satisfy (5.1), when restricting to the case of one Markov process, cf. [4], Assumptions A, C(3), [10] Assumption 3.2 and [9] Assumptions 2.1 and 2.2.

Let Assumption 1.1 be satisfied, and let  $V : \mathbf{S} \rightarrow [1, \infty)$  be a  $c$ -drift moment function. Further assume that there exists a function  $W' : \mathbf{S} \rightarrow \mathbf{R}_+$ , and constants  $b', c'$  and  $M'$  such that for  $x \in \mathbf{S}$

$$q_x V(x) \leq M' W'(x) \quad (5.5)$$

$$\sum_y q_{xy} W'(y) \leq c' W'(x) + b'. \quad (5.6)$$

Then the Kolmogorov forward integral equation (5.1) holds for any function  $f \in \ell^\infty(\mathbf{S}, V)$ .

Note that we have slightly adapted the formulation of the conditions from these works. The proof of the statement by the way has not been provided explicitly, as far as we know. The condition of a  $V$ -moment drift function is not necessary, as we have seen. Moreover, the condition transforms to Condition 4.4 for the  $V$ -transformation. However, Example 4.5 shows that this is stronger than condition 4.1 for the  $V$ -transformation, and as we have just proved, the latter is sufficient to guarantee non-explosiveness of the  $V$ -transformation and the validity of the Kolmogorov forward integral equation for all  $V$ -bounded functions.

A second comment concerns a general remark on the validity of the Kolmogorov forward integral equation. In [3] Appendix C3 the validity is asserted for functions  $f : \mathbf{S} \rightarrow \mathbf{R}$  with  $E_x |f|(X_t) < \infty$  and

$\mathbb{E}_x \int_0^t |Qf(X_s)| ds < \infty$  for all  $t \geq 0$  and  $x \in \mathbf{S}$ . As we have seen, this is not true for  $c$ -drift functions  $f$ , for which  $X^f$  is explosive. An example has been provided in [12].

In [2] Theorem 9.2.2 a similar statement has been made. There the Markov process itself is required to be non-explosive, which is not necessary. The theorem in that book states that the condition  $\mathbb{E}_x \int_0^t q_{X_s} |f|(X(s)) ds < \infty$  is equivalent to  $\mathbb{E}_x |f|(X_t) < \infty$ . The same case mentioned above, shows that the second property does not imply the first. If it would imply it, then the Kolmogorov forward integral equation would hold, but we know that it does not.

Clearly some delicacies are involved in the study of countable state Markov processes.

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