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A Bound on Powers of Linear Operators, With Relevance to Numerical Stability

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Abstract—In this note, we formulate a theorem giving bounds on the powers of linear operators, in a general Banach space setting. The relevance of the theorem is illustrated by applying it to the Crank-Nicholson method for the numerical solution of the heat equation. This application yields a stability estimate in the maximum norm which amounts to an improvement over a well-known result of Serdjukova [1].

Keywords—Power bounded operators, Resolvent condition, Numerical analysis, Stability analysis, Heat equation, Finite-difference scheme, Crank-Nicholson method.

1. INTRODUCTION

Let T denote a given bounded linear operator from a Banach space into itself. In the literature, general upper bounds on the norm of T^n (for $n \geq 1$) were established, under various conditions on the resolvent of T, see, e.g., [2–7]. In this note, we present a new general bound on the norm of T^n and we apply this bound in deriving some actual stability estimates. The estimates thus obtained do not follow from earlier general bounds on T^n known to the authors.

In Section 2, we present Theorem 2.1, which contains our new bound. The proof of the theorem relies, among other things, on an argument used earlier in [7].

In Section 3, we apply Theorem 2.1 to the well-known Crank-Nicholson method for the numerical solution of the heat equation. First, in Section 3.1, we introduce some notations and we review results from the literature. Next, in Section 3.2, we combine Theorem 2.1 and material from Section 3.1. In this way, we easily arrive at explicit stability estimates, in the maximum norm, for the Crank-Nicholson method. These estimates are sharper than a well-known stability result obtained in [1].

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2. A GENERAL UPPER BOUND ON $\|T^n\|$

In this section, X denotes an arbitrary complex Banach space, with $X \neq \{0\}$. By $\|\cdot\|$ we denote the corresponding operator norm, and by I the identity operator.

Let T be a bounded linear operator from X into itself. We shall determine an upper bound for $\|T^n\|$, under the assumption that the spectrum $\sigma[T]$ of T satisfies

$$\sigma[T] \subset \{z : |z| \leq 1\}. \tag{2.1a}$$

In addition to (2.1a), we shall make the following assumptions (2.1b)–(2.1d), in which fixed integers $m \geq 1$, $k \geq 0$, and $p(i) \geq 0$ occur. As yet we do not specify these integers any further; in applications of the subsequent Theorem 2.1, the actual values of m, k, $p(i)$ follow from the context.

In the following, $H_1, H_2, \ldots, H_m$ will denote operator-valued holomorphic functions such that, for all complex z with $|z| < 1$,

$$(I - zT)^{-k-1} = H_1(z) + H_2(z) + \cdots + H_m(z). \tag{2.1b}$$

Further, f_i and g_i will denote complex-valued holomorphic functions such that, for $i = 1, 2, \ldots, m$ and all z with $|z| < 1$,

$$\|H_i(z)\| \leq \left| z^{p(i)} f_i(z)\, g_i(z) \right|, \tag{2.1c}$$

$$f_i(z) = a_0(i) + a_1(i)z + a_2(i)z^2 + \cdots, \qquad g_i(z) = b_0(i) + b_1(i)z + b_2(i)z^2 + \cdots. \tag{2.1d}$$

The following theorem gives a bound on $\|T^n\|$ in terms of the coefficients $a_j(i)$ and $b_j(i)$. We note that, for the special case where $m = k = 1$ and $f_1 = g_1$, the theorem may be viewed as a variant to a result given in [7].

We shall use the convention throughout that $\sum_{j=r}^{s} \cdots = 0$ whenever $r > s$.

THEOREM 2.1. *Let m, k, $p(i)$ be any fixed integers with $m \geq 1$, $k \geq 0$, $p(i) \geq 0$. Assume (2.1), and let $n \geq 1$. Then*

$$\|T^n\| \leq \frac{\gamma_1 + \gamma_2 + \cdots + \gamma_m}{\binom{n+k}{k}}, \tag{2.2a}$$

where all terms γ_i can be chosen according to

$$\gamma_i = \left(\sum_{j=0}^{n-p(i)} |a_j(i)|^2 \right)^{1/2} \left(\sum_{j=0}^{n-p(i)} |b_j(i)|^2 \right)^{1/2}. \tag{2.2b}$$

Moreover, for any values i for which a constant operator C_i exists with $H_i(z) = z^{p(i)} f_i(z) g_i(z) C_i$ (for $|z| < 1$), the terms γ_i can also be chosen according to the (more favourable) formula

$$\gamma_i = \left| \sum_{j=0}^{n-p(i)} a_j(i)\, b_{n-p(i)-j}(i) \right|. \tag{2.2c}$$

PROOF.

1. In view of (2.1a), we have $(I - zT)^{-k-1} = \sum_{n=0}^{\infty} \binom{n+k}{k} z^n T^n$ (for $|z| < 1$). Therefore,

$$\binom{n+k}{k} T^n = \frac{1}{2\pi i} \oint z^{-n-1} (I - zT)^{-k-1}\, dz = \sum_{i=1}^{m} \frac{1}{2\pi i} \oint z^{-n-1} H_i(z)\, dz,$$

where the integration is along any positively oriented circle $|z| = r$, with $0 < r < 1$. We thus obtain

$$\|T^n\| \leq \frac{1}{\binom{n+k}{k}} \sum_{i=1}^{m} \|J_i\|, \qquad \text{with } J_i = \frac{1}{2\pi\mathrm{i}} \oint z^{-n-1} H_i(z)\, \mathrm{d}z. \tag{2.3}$$

2. Assume, with no loss of generality, that none of the functions f_i or g_i vanishes identically for $|z| < 1$.

   We shall use (below in Part 3 of the Proof) that J_i, as defined in (2.3), satisfies

$$J_i = \frac{1}{2\pi\mathrm{i}} \oint z^{-n-1} \frac{\phi_i(z)\psi_i(z)}{f_i(z)g_i(z)} H_i(z)\, \mathrm{d}z, \tag{2.4a}$$

   where

$$\phi_i(z) = \sum_{j=0}^{n-p(i)} a_j(i) z^j, \qquad \psi_i(z) = \sum_{j=0}^{n-p(i)} b_j(i) z^j. \tag{2.4b}$$

   In order to prove (2.4), we note first that, in view of (2.1c), the operator-valued function $E_i(z) = [z^{p(i)} f_i(z) g_i(z)]^{-1} H_i(z)$ (after elimination of removable singularities) is holomorphic for $|z| < 1$. Further, it is easy to see that the function $f_i g_i - \phi_i \psi_i$ has a power series of the form $f_i(z)g_i(z) - \phi_i(z)\psi_i(z) = c_0(i) + c_1(i)z + c_2(i)z^2 + \cdots$, with $c_j(i) = 0$ whenever $0 \leq j \leq n - p(i)$. Therefore, denoting the expression in the right-hand member of (2.4a) by K_i, we obtain

$$J_i - K_i = \frac{1}{2\pi\mathrm{i}} \oint z^{-n-1+p(i)} \left(c_{n-p(i)+1}(i)\, z^{n-p(i)+1} + c_{n-p(i)+2}(i)\, z^{n-p(i)+2} + \cdots \right) E_i(z)\, \mathrm{d}z$$

   (where we have used the definition $c_j(i) = 0$ whenever $j < 0$). Since the last integral equals zero, the proof of (2.4) is complete.

3. Using (2.4a), we obtain in view of (2.1c) and the Schwarz inequality

$$\|J_i\| \leq \frac{r^{p(i)-n-1}}{2\pi} \oint |\phi_i(z)|\, |\psi_i(z)|\, |\mathrm{d}z| \leq \frac{r^{p(i)-n-1}}{2\pi} \sqrt{\oint |\phi_i(z)|^2 |\mathrm{d}z|} \cdot \sqrt{\oint |\psi_i(z)|^2\, |\mathrm{d}z|}.$$

   By using (2.4b) and Parseval's formula, we arrive at

$$\|J_i\| \leq r^{p(i)-n} \sqrt{\sum_{j=0}^{n-p(i)} |a_j(i)|^2\, r^{2j}} \cdot \sqrt{\sum_{j=0}^{n-p(i)} |b_j(i)|^2\, r^{2j}}.$$

   Let γ_i be defined by (2.2b). By letting $r \to 1$ in the last inequality, we obtain $\|J_i\| \leq \gamma_i$. In view of (2.3), we arrive at (2.2a).

4. Consider now the case where, for some index i, we have $H_i(z) = z^{p(i)} f_i(z)\, g_i(z)\, C_i$. From (2.4), we obtain

$$J_i = \frac{1}{2\pi\mathrm{i}} \oint z^{-n-1+p(i)} \left(\sum_{l=0}^{2(n-p(i))} c_l(i) z^l \right) \mathrm{d}z \cdot C_i,$$

   with coefficients $c_l(i)$ satisfying $c_l(i) = \sum_{j=0}^{l} a_j(i)\, b_{l-j}(i)$ (for $0 \leq l \leq n - p(i)$). Consequently, $J_i = (\sum_{j=0}^{n-p(i)} a_j(i)\, b_{n-p(i)-j}(i)) \cdot C_i$. This implies that $\|J_i\| \leq \gamma_i$, where γ_i is defined by (2.2c). In view of (2.3), the proof of the theorem is complete. ∎

REMARK 2.2. Theorem 2.1 would no longer be true if, with regard to (2.2c), the condition $H_i(z) = z^{p(i)} f_i(z) g_i(z) C_i$ (for $|z| < 1$) would be omitted. This can be seen from the following counterexample.

Let $T = I$. We have $(I - zT)^{-1} = (1 + z + z^2 + \cdots)I = f_1(z)\, g_1(z)\, E_1(z)$, with $f_1(z) = 2/(1-z)$, $g_1(z) = 1/(1+z)$, $E_1(z) = [(1+z)/2]I$. Condition (2.1) is fulfilled with $k = 0$, $m = 1$, $p(1) = 0$, and $a_j(1) = 2$, $b_j(1) = (-1)^j$. The value γ_i defined by (2.2c) (with $i = 1$) is equal to zero for $n = 1, 3, 5, \ldots$. But, for all n, we have $\|T^n\| = 1$.

REMARK 2.3. Let $k \geq 0$ and $0 \leq l \leq m$. Assume (2.1) and let $H_i(z) = z^{p(i)} f_i(z) g_i(z) E_i(z)$. In case $E_i(z)$ is an operator-valued polynomial with respect to the variable z, we denote its degree by $q(i)$; otherwise we define $q(i) = \infty$. It can be proved that, for $n \geq 1$, the bound (2.2a) is valid, if we define γ_i, for $l < i \leq m$, by (2.2b) and, for $1 \leq i \leq l$, by

$$\gamma_i = \left(\sum_{\nu = p(i)}^{p(i)+q(i)} \left| \sum_{j=0}^{n-\nu} a_j(i) b_{n-\nu-j}(i) \right|^2 \right)^{1/2}. \tag{2.5}$$

Clearly, for $q(i) = 0$, formula (2.5) coincides with (2.2c), which is an improvement over (2.2b). For $q(i) \geq 1$, however, there are cases where (2.5) gives a larger value than (2.2b).

3. AN APPLICATION OF THEOREM 2.1 TO THE CRANK-NICHOLSON METHOD

3.1. Stability of the Crank-Nicholson Scheme

Consider the initial-boundary value problem for the one-dimensional heat equation

$$u_t(x,t) = u_{xx}(x,t), \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = f(x),$$

where $0 \leq x \leq 1$, $t \geq 0$, and f denotes a given real function.

The well-known Crank-Nicholson method, for approximating the solution $u(x,t)$ to the above problem, consists in the application of the following finite-difference scheme:

$$\frac{v_{m,n} - v_{m,n-1}}{\Delta t} = \frac{1}{2}\left[\frac{v_{m-1,n} - 2v_{m,n} + v_{m+1,n}}{(\Delta x)^2} + \frac{v_{m-1,n-1} - 2v_{m,n-1} + v_{m+1,n-1}}{(\Delta x)^2} \right].$$

Here $\Delta t > 0$, $\Delta x = 1/(M+1)$, $m = 1, 2, \ldots, M$, and $n = 1, 2, 3, \ldots$. Choosing $v_{0,n} = v_{M+1,n} = 0$ and $v_{m,0} = f(m\Delta x)$, we see that $v_{m,n}$ approximates $u(x,t)$ for $x = m\Delta x$ and $t = n\Delta t$.

In order to rewrite the Crank-Nicholson scheme in a compact form, we put $\alpha = \Delta t/(\Delta x)^2$, and we define the $M \times M$ matrices A and T by

$$A = A(M,\alpha) = \frac{\alpha}{2}\,\text{tridiag}(1,-2,1), \qquad T = T(M,\alpha) = (I + A)(I - A)^{-1}. \tag{3.1}$$

Further, we denote by v_n the column vector with components $v_{1,n}, v_{2,n}, \ldots, v_{M,n}$. Clearly, the Crank-Nicholson scheme is equivalent to

$$v_n = T v_{n-1}, \qquad n = 1, 2, 3, \ldots. \tag{3.2}$$

In dealing with the stability of the numerical process (3.2), we shall use the norms

$$\|x\|_p = \left(\sum_{m=1}^{M} |\xi_m|^p \right)^{1/p}, \qquad \text{for } 1 \leq p < \infty,$$

$$\|x\|_p = \max_{1 \leq m \leq M} |\xi_m|, \qquad \text{for } p = \infty.$$

Here x denotes any vector in $\mathbb{C}^M$, with components ξ_m. Further, for arbitrary $M \times M$ matrices A, we shall use the notation

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

In the literature on the stability of the Crank-Nicholson process (3.2), much attention was paid to establishing upper bounds on $\sup_{n \geq 1} \|T(M,\alpha)^n\|_p$ for various values of p and α; see, e.g., [8–10] and the references therein.

It has been known for a long time that $\|T(M,\alpha)^n\|_2 \leq 1$ (for all $\alpha > 0$, $M \geq 1$, $n \geq 1$). But, for the case of the important norm $\|\cdot\|_\infty$ the situation is considerably less simple. The following well-known general bound on $\|T(M,\alpha)^n\|_\infty$ was given in [1]

$$\|T(M,\alpha)^n\|_\infty \leq 23, \qquad \text{for all } \alpha > 0, \quad M \geq 1, \quad n \geq 1. \tag{3.3}$$

In Section 3.2, we shall see that this stability estimate can easily be improved by applying Theorem 2.1 in combination with the subsequent lemma. In all of the following, A and T will denote the matrices defined by (3.1).

LEMMA 3.1.

a. *We have* $\sigma[A] \subset (-\infty, 0)$ *and* $\sigma[T] \subset (-1,1)$.
b. *Let* $|z| < 1$. *Then*

$$(I - zT)^{-1} = \frac{1}{1+z} I + \frac{2z}{(1+z)(1-z)} [I - \varphi(z)A]^{-1},$$

where $\varphi(z) = (1+z)/(1-z)$ *with* $\mathrm{Re}[\varphi(z)] > 0$.
c. *Let* $|z| < 1$, *and* $\varphi(z)$ *as above. Then* $\|[I - \varphi(z)A]^{-1}\|_\infty \leq \sqrt{2}$.

PROOF.

a. It is well known that the eigenvalues of A are real and negative, see, e.g., [11, p. 52]. In view of (3.1) we conclude, by the spectral mapping theorem, that the eigenvalues of T lie in the interval $(-1,1)$.
b. A straightforward calculation, using (3.1) and Part a of the lemma, proves Part b.
c. It was proved (implicitly) in [12, pp. 59–60] that $\|(\zeta I - A)^{-1}\|_\infty \leq \sqrt{2}/|\zeta|$ (for $\mathrm{Re}\,\zeta > 0$). Hence, $\|(I - \zeta A)^{-1}\|_\infty \leq \sqrt{2}$ (for $\mathrm{Re}\,\zeta > 0$). Since $\mathrm{Re}[\varphi(z)] > 0$, the proof is complete. ∎

3.2. Estimating $\|T^n\|$ for the Crank-Nicholson Operator T

In view of Lemma 3.1, it is natural to apply the general Theorem 2.1 to the Crank-Nicholson operator T with $X = \mathbb{C}^M$, $\|\cdot\| = \|\cdot\|_\infty$, $k \geq 0$, and $m = k+2$. The authors found that such applications, with $k \geq 2$, yield larger upper bounds for $\|T^n\|_\infty$ than can be obtained with $k = 0$ or $k = 1$. For that reason, we confine ourselves in the following to the two last-mentioned values of k.

We note that our actual stability result presented below cannot be obtained by combining Lemma 3.1 and earlier general bounds as given, e.g., in [2–7].

3.2.1. Applying Theorem 2.1, with $k = 0$, to T defined by (3.1)

We use the same notations as in Lemma 3.1, and define

$$H_1(z) = \frac{1}{1+z} I, \qquad H_2(z) = \frac{2z}{(1+z)(1-z)} [I - \varphi(z)A]^{-1},$$

$$f_1(z) = \frac{1}{1+z} = 1 - z + z^2 + \cdots, \qquad g_1(z) = 1,$$

$$f_2(z) = g_2(z) = \left[\frac{c}{(1+z)(1-z)}\right]^{1/2} = \sqrt{c}\left[1 - \binom{-\frac{1}{2}}{1} z^2 + \binom{-\frac{1}{2}}{2} z^4 - \cdots\right],$$

where $c = 2\sqrt{2}$. Furthermore, we define $p(1) = 0$, $p(2) = 1$. In view of Lemma 3.1, the Crank-Nicholson operator T now fulfills conditions (2.1a)–(2.1d) with $k = 0$, $m = 2$ and $\|\cdot\| = \|\cdot\|_\infty$.

Since $H_1(z) = z^{p(1)} f_1(z) g_1(z) I$, an application of Theorem 2.1 with γ_1 defined by (2.2c) and γ_2 defined by (2.2b) yields, for all $n \geq 1$, the upper bound

$$\|T^n\|_\infty \leq c_n. \tag{3.4}$$

Here

$$c_n = 1 + 2\sqrt{2} \sum_{j=0}^{N} \begin{pmatrix} -\frac{1}{2} \\ j \end{pmatrix}^2, \tag{3.5a}$$

where

$$\begin{aligned} N &= \frac{(n-1)}{2}, \qquad \text{for odd } n \geq 1, \\ N &= \frac{(n-2)}{2}, \qquad \text{for even } n \geq 2. \end{aligned} \tag{3.5b}$$

Some values of c_n (rounded to four decimal places) are listed in Table 3.1.

Table 3.1. Some values of c_n.

n	1	2	3	4	5	6	7	8	9	10
c_n	3.828	3.828	4.536	4.536	4.933	4.933	5.209	5.209	5.421	5.421

Clearly, for small $n \geq 1$, the values c_n are of moderate size. But, it can be proved that $c_n \to \infty$ (for $n \to \infty$). This unpleasant behaviour is not present in the upper bounds which we shall obtain below in Section 3.2.2 with $k = 1$.

3.2.2. Applying Theorem 2.1, with $k = 1$, to T defined by (3.1)

We again use the same notations as in Lemma 3.1, and define

$$H_1(z) = \left[\frac{1}{(1+z)} I\right]^2, \qquad H_2(z) = \left[\frac{z}{(1+z)(1-z)} R(z)\right]^2, \qquad H_3(z) = \frac{2z}{(1+z)^2(1-z)} R(z),$$

where $R(z) = 2[I - \varphi(z)A]^{-1}$. Further, we put

$$\begin{aligned} f_1(z) &= g_1(z) = (1+z)^{-1} = 1 - z + z^2 + \cdots, \\ f_2(z) &= g_2(z) = c(1-z^2)^{-1} = c[1 + z^2 + z^4 + \cdots], \\ f_3(z) &= 2 f_1(z), \qquad g_3(z) = g_2(z), \end{aligned}$$

where $c = 2\sqrt{2}$. Finally, we define $p(1) = 0$, $p(2) = 2$, $p(3) = 1$. In view of Lemma 3.1, the Crank-Nicholson operator T now satisfies conditions (2.1a)–(2.1d) with $k = 1$, $m = 3$, and $\|\cdot\| = \|\cdot\|_\infty$.

Defining γ_1, γ_2, and γ_3 according to (2.2b), we easily find $\gamma_1 = n + 1$, $\gamma_2 = c^2 n/2$ (for even $n \geq 2$), $\gamma_2 = c^2(n-1)/2$ (for odd $n \geq 1$), and $\gamma_3 = \sqrt{2}cn$ (for even $n \geq 2$), $\gamma_3 = \sqrt{2}c\sqrt{n(n+1)}$ (for odd $n \geq 1$). By virtue of Theorem 2.1, we thus obtain

$$\|T^n\|_\infty \leq 1 + \frac{n}{n+1} \cdot 8 < 9, \qquad \text{for } n \geq 1. \tag{3.6}$$

Since T is a real symmetric matrix, we have $\|T^n\|_1 = \|T^n\|_\infty$. Therefore, an application of the Riesz convexity theorem (see, e.g., [13, Chapter VI]) yields $\|T^n\|_p \leq \|T^n\|_\infty$ (for $1 \leq p < \infty$). In view of (3.4),(3.6), we thus arrive at the following.

CONCLUSION 3.2. *Let $1 \leq p \leq \infty$. Then for any ratio $\alpha = \Delta t/(\Delta x)^2 > 0$ and integers $M \geq 1$, $n \geq 1$, the Crank-Nicholson operator $T(M, \alpha)$ (see (3.1)) allows the stability estimates*

$$\|T(M,\alpha)^n\|_p \leq 1 + \frac{n}{n+1} \cdot 8 < 9 \qquad \text{and} \qquad \|T(M,\alpha)^n\|_p \leq c_n.$$

Here c_n is defined by (3.5), and some values are given in Table 3.1.

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