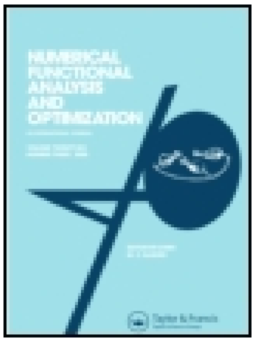

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A NOTE ABOUT RITT'S CONDITION, RELATED RESOLVENT CONDITIONS AND POWER BOUNDED OPERATORS

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Abstract. Ritt [17] formulated a condition on the resolvent of a bounded linear operator T, under which he concluded that $n^{-1}\|T^n\| \to 0$ when $n \to \infty$. In this note we show that Ritt's condition can be related to various other resolvent conditions, some of which were considered in the (recent) literature. Furthermore, we prove that Ritt's condition, as well as several akin conditions, actually imply that $\sup\{\|T^n\| : n \geq 1\} < \infty$. Finally, we indicate that the resolvent conditions under consideration have a relation to numerical methods for solving initial value problems.

1. Introduction

In the following T denotes a bounded linear operator from an arbitrary non-zero complex Banach space X into itself. We denote the spectrum of T by $\sigma[T]$, and the resolvent set by $\rho[T] = \mathbb{C}\backslash\sigma[T]$. We shall say that T is *power bounded* if $\sup\|T^n\| < \infty$, where the supremum is over all positive integers n. The identity operator in X will be denoted by I.

In the stability analysis of numerical processes for solving initial value problems one is often faced with the problem of estimating $\|T^n\|$ for $n \geq 1$; stable processes are distinguished by the property that $\|T^n\|$ is of moderate size. Usually the dimension of X is finite. But, when T denotes the numerical solution operator, corresponding to a given initial-boundary value problem, one has to pay special attention to the situation where the dimension tends to infinity (see Section 4 and e.g. [3], [22], [24], [26]). As a result, estimates for $||T^n||$ in a general Banach space setting are very relevant to numerical analysis.

An early contribution to estimating $||T^n||$ in the context of arbitrary Banach spaces was made by Ritt [17]. He dealt with the following condition (1.1), in which M and η denote given constants.

$$\begin{aligned} &\sigma[T] \subset \{\zeta: |\zeta|<1 \text{ or } \zeta=1\}, \quad \text{and} \\ &||(\zeta-1)\cdot(\zeta I-T)^{-1}|| \leq M \\ &\text{for all } \zeta \in \mathbb{C} \text{ with } |\zeta| \geq 1,\ 0<|\zeta-1| \leq \eta. \end{aligned} \tag{1.1}$$

Ritt [17] proved that, if T obeys the hypothesis (1.1), one has

$$\lim_{n\to\infty} \frac{||T^n||}{n} = 0\,. \tag{1.2}$$

One of the purposes of the present note is to improve Ritt's conclusion (1.2) and to relate his condition (1.1) to the (recent) literature in the fields of operator theory and numerical analysis.

In Section 2 of this note we shall review a condition of Tadmor [25] as well as various other conditions on the growth of the resolvent $(\zeta I - T)^{-1}$, considered in the literature. We shall analyse what relations these conditions have to each other as well as to (1.1). Tadmor's condition will be shown to be essentially equivalent to (1.1).

In Section 3.1 estimates of $||T^n||$ will be reviewed which were obtained in the literature under the resolvent conditions discussed in Section 2. Furthermore, Ritt's conclusion (1.2) will be improved substantially by combining the results obtained in Section 2 (Theorems 2.1 and 2.3) with material from [21]. In this manner, condition (1.1) will be shown to imply power boundedness of T. In Section 3.2 we shall give an alternative, very short proof of power boundedness, without using any results from the literature. In Section 3.3 we shall specify generalized versions of Tadmor's condition under which we can still give a direct and short proof of power boundedness.

In Section 4 we shall shortly relate the resolvent conditions and estimates of $||T^n||$, discussed in the foregoing sections, to the stability of numerical processes for solving initial-boundary value problems.

We conclude this introduction by formulating a simple lemma, which we shall use repeatedly; its proof is easy (see e.g. Dunford & Schwartz [5], or Dowson [4]).

Lemma 1.1. *The function $R(\zeta) = (\zeta I - T)^{-1}$ is continuous on $\rho[T]$. Furthermore, if $\xi \in \rho[T]$ and $\alpha = |\zeta - \xi| \cdot ||R(\xi)|| < 1$, then also $\zeta \in \rho[T]$ and $||R(\zeta)|| \leq \frac{||R(\xi)||}{1-\alpha}$.*

2. Resolvent conditions related to (1.1)

2.1. The resolvent conditions of Tadmor and Kreiss

Tadmor [25] discussed the growth of $\|T^n\|$ (for $n \geq 1$) under the following assumption (2.1), where L denotes a given constant.

$$(2.1) \qquad \sigma[T] \subset \{\zeta : |\zeta| \leq 1\}, \quad \text{and}$$
$$\|(\zeta I - T)^{-1}\| \leq \frac{L}{|\zeta - 1|} \quad \text{for all } \zeta \in \mathbb{C} \text{ with } |\zeta| > 1.$$

The theorem below shows that Ritt's condition (1.1) and Tadmor's condition (2.1) can be regarded to be equivalent.

Theorem 2.1. *There exist constants M and $\eta > 0$ such that (1.1) holds, if and only if there is an L such that (2.1) is valid.*

Proof. **1.** Assume (1.1). Since the function $F(\zeta) = |\zeta - 1| \cdot \|R(\zeta)\|$ is continuous on $\rho[T]$, and $F(\zeta) \to 1$ (for $|\zeta| \to \infty$), there is a finite constant K such that

$$F(\zeta) \leq K \quad \text{for all } \zeta \in \mathbb{C} \text{ with } |\zeta| \geq 1,\ |\zeta - 1| \geq \eta .$$

In view of (1.1), we arrive at (2.1) with $L = \max\{K, M\}$.

2. Assume (2.1), and let ζ be given with $|\zeta| = 1$, $\zeta \neq 1$. Choosing ξ, with $|\xi| > 1$, sufficiently close to ζ we have by using the notations of Lemma 1.1

$$\alpha = |\zeta - \xi| \cdot \|R(\xi)\| \leq |\zeta - \xi| \cdot \frac{L}{|\xi - 1|} < 1 .$$

From the lemma we conclude that $\zeta \in \rho[T]$ with $|\xi - 1| \cdot (1 - \alpha)\|R(\zeta)\| \leq L$. By letting $\xi \to \zeta$, it follows that $|\zeta - 1| \cdot \|R(\zeta)\| \leq L$.

Thus, we have proved (1.1) with $M = L$ and any $\eta > 0$. □

Kreiss [8] studied the size of $\|T^n\|$ (for $n \geq 1$) in his stability analysis of numerical processes for the solution of initial value problems. For the case where X is of finite dimension, he dealt with the assumption

$$(2.2) \qquad \sigma[T] \subset \{\zeta : |\zeta| \leq 1\}, \quad \text{and}$$
$$\|(\zeta I - T)^{-1}\| \leq \frac{L}{|\zeta| - 1} \quad \text{for all } \zeta \in \mathbb{C} \text{ with } |\zeta| > 1.$$

Since $|\zeta| - 1 \leq |\zeta - 1|$, condition (2.1) implies (2.2). But, the obvious counterexample $T = -I$ shows that (2.2) does not imply (2.1) — not even with a larger constant L in (2.1) than in (2.2). We conclude that assumption (2.1) is essentially stronger than (2.2).

Below, in Section 2.2, we shall formulate variants to the conditions of Tadmor and Kreiss which, notwithstanding our last conclusion, will turn out to be equivalent to each other.

Remark 2.2. Condition (2.1), as well as (2.2), implies that $L \geq 1$. This can be seen from the resolvent inequalities by letting $\zeta \to \infty$.

2.2. Variants to the conditions of Tadmor and Kreiss

We start by giving some definitions needed in formulating the subsequent variants to (2.1) and (2.2).

For $\xi \in \mathbb{C}$, $\xi \neq 0$, we denote by $\mathrm{Arg}(\xi)$ the *argument* with $-\pi < \mathrm{Arg}(\xi) \leq \pi$, and we put $\mathrm{Arg}(0) = 0$. We define the set

$$W_\delta = \{\xi : \ \xi \in \mathbb{C} \ \text{ with } |\xi| \leq 1 - \delta \cdot |\mathrm{Arg}\,(\xi)|\} ,$$

where δ denotes a constant satisfying

$$0 \leq \delta \leq 1/\pi.$$

For $\zeta \in \mathbb{C}$ and $V \subset \mathbb{C}$ we denote the *distance* from ζ to V by

$$\mathrm{d}(\zeta, V) = \inf\{|\zeta - \xi| : \ \xi \in V\} .$$

We shall deal with the resolvent condition

$$\text{(2.3)} \qquad \begin{aligned} &\sigma[T] \ \subset \ W_\delta, \quad \text{and} \\ &\|(\zeta I - T)^{-1}\| \ \leq \ \frac{K}{|\zeta - 1|} \quad \text{for all } \zeta \in \mathbb{C}\backslash W_\delta \, , \end{aligned}$$

as well as with the condition

$$\text{(2.4)} \qquad \begin{aligned} &\sigma[T] \ \subset \ W_\delta, \quad \text{and} \\ &\|(\zeta I - T)^{-1}\| \ \leq \ \frac{K}{\mathrm{d}(\zeta, W_\delta)} \quad \text{for all } \zeta \in \mathbb{C}\backslash W_\delta \, . \end{aligned}$$

The latter condition has been taken from Spijker & Straetemans [21].

Note that if $\delta = 0$ and $K = L$, the conditions (2.3), (2.4) neatly reduce to (2.1) and (2.2), respectively. For $\delta > 0$, the conditions (2.3), (2.4) seem to be essentially stronger than (2.1), (2.2), respectively. Further, (2.3) may seem to be essentially stronger than (2.4). However, the following theorem shows that, when disregarding the precise magnitude of K, L and $\delta > 0$, the three conditions (2.1), (2.3) and (2.4) are equivalent to each other.

Theorem 2.3.
(a) *For $0 < \delta \leq 1/\pi$ condition (2.3) implies (2.4).*
(b) *For $0 < \delta \leq 1/\pi$ condition (2.4) implies (2.1) with $L = c \cdot K$, where $c = \dfrac{1 + \epsilon}{(1 - \epsilon\pi)\epsilon}$ and $\epsilon = \min\{\delta, \sqrt{1 + 1/\pi} - 1\}$.*
(c) *Condition (2.1) implies (2.3) with $\delta = 1/(\pi\, L)$, $K = 3L$.*

Proof. **1.** Part (a) of the Theorem follows from the inequality $\mathrm{d}(\zeta, W_\delta) \leq |\zeta - 1|$.

2. In order to prove part (b), we assume that $0 < \delta \leq 1/\pi$ and we consider

$$\zeta = |\zeta|\mathrm{e}^{\mathrm{i}\theta}\,, \text{ with } |\zeta| > 1 \text{ and } -\pi < \theta \leq \pi\,.$$

We shall derive a lower bound for $\mathrm{d}(\zeta, W_\delta)$ in terms of $|\zeta - 1|$.

It can be shown (see Spijker & Straetemans [21, Lemma 5.1]) that, for any ϵ with $0 \leq \epsilon \leq 1/\pi$,

$$\mathrm{d}(\zeta, W_\epsilon) \geq \frac{1 - \epsilon\pi}{1 + \epsilon}(|\zeta| - 1 + \epsilon|\theta|)\,.$$

Since $|\zeta - 1| \leq |\zeta - \mathrm{e}^{\mathrm{i}\theta}| + |\mathrm{e}^{\mathrm{i}\theta} - 1| \leq |\zeta| - 1 + |\theta|$, it follows that $|\zeta| - 1 + \epsilon|\theta| \geq \epsilon|\zeta - 1|$. Defining $f(\epsilon) = (1 - \epsilon\pi)\epsilon/(1 + \epsilon)$, we thus have

$$\mathrm{d}(\zeta, W_\epsilon) \geq f(\epsilon) \cdot |\zeta - 1| \quad (\text{for } 0 \leq \epsilon \leq 1/\pi)\,. \tag{2.5}$$

The function f attains its maximum value on the interval $[0, 1/\pi]$ at $\epsilon = \sqrt{1 + 1/\pi} - 1$. For this reason we now choose

$$\epsilon = \min\{\delta, \sqrt{1 + 1/\pi} - 1\}\,.$$

Using (2.5) with this choice for ϵ, we thus obtain

$$\mathrm{d}(\zeta, W_\delta) \geq \frac{|\zeta - 1|}{c}\,, \text{ where } c = \frac{1 + \epsilon}{(1 - \epsilon\pi)\epsilon}\,.$$

From the last inequality we see that part (b) of the theorem is true.

3. Assume (2.1). In order to prove part (c) we shall first show that the relations

$$\zeta = |\zeta|\mathrm{e}^{\mathrm{i}\theta}, \quad 0 \leq (1 - |\zeta|)2L < |\mathrm{e}^{\mathrm{i}\theta} - 1| \tag{2.6a}$$

imply that

$$\zeta \in \rho[T], \quad \|(\zeta I - T)^{-1}\| \leq 2L|\mathrm{e}^{\mathrm{i}\theta} - 1|^{-1}. \tag{2.6b}$$

Assume (2.6a), and let $\xi = |\xi|\mathrm{e}^{\mathrm{i}\theta}$ with $|\xi| > 1$. We have

$$|\zeta - \xi| \cdot \|(\xi I - T)^{-1}\| \leq (|\xi| - |\zeta|) \cdot L|\xi - 1|^{-1}.$$

For $\xi \to e^{i\theta}$, the right-hand member of this inequality tends to

$$(1-|\zeta|)\cdot L|e^{i\theta}-1|^{-1} < 1/2\,.$$

Applying Lemma 1.1, we see that (2.6b) holds.

Still assuming (2.6a) we have, in view of Remark 2.2,

$$|\zeta-1| \le |e^{i\theta}-1| + |e^{i\theta}-\zeta| = |e^{i\theta}-1| + 1 - |\zeta| \le \frac{3}{2}|e^{i\theta}-1|.$$

In view of (2.6b) we conclude that (2.6a) implies

$$\text{(2.6c)} \qquad \zeta \in \rho[T], \quad \|(\zeta I - T)^{-1}\| \le 3L\cdot|\zeta-1|^{-1}.$$

Let $\delta = 1/(\pi L)$, and $\zeta \in \mathbb{C}\backslash W_\delta$ with $|\zeta| \le 1$. In order to complete the proof of the theorem it is sufficient to establish (2.6c).

We have, with $\theta = \mathrm{Arg}(\zeta)$,

$$0 \le 1 - |\zeta| < \frac{1}{\pi L}|\theta| \le \frac{1}{2L}|e^{i\theta}-1|\,.$$

Hence (2.6a) is fulfilled and therefore (2.6c) holds. □

3. Bounding $\|T^n\|$ (for $n \ge 1$)

3.1. Results from the literature

In the literature various estimates for $\|T^n\|$ are available under the resolvent conditions considered in Section 2.1. Below we give a short review.

According to Tadmor [25], condition (2.1) implies the estimate

$$\text{(3.1)} \qquad \sup_{n\ge 2} \frac{\|T^n\|}{\log n} < \infty\,.$$

In view of Theorem 2.1, this estimate can be viewed as an improvement over Ritt's conclusion (1.2) under assumption (1.1).

In the literature much attention was paid to estimating $\|T^n\|$ under the condition (2.2) for the case where X is of finite dimension k; see Kreiss [8], Morton [11], Tadmor [24], LeVeque & Trefethen [10], Spijker [20], Dorsselaer et al [3], Strikwerda & Wade [23]. It is known that (2.2) implies the upper bound

$$\|T^n\| \le eLk \quad (n \ge 1)\,.$$

Further, without making any assumption about the dimension of X, condition (2.2) is known to imply

$$\|T^n\| \leq eL(n+1) \quad (n \geq 1).$$

The last two upper bounds are known to be essentially sharp – power boundedness cannot be proved in general (cf. Kraaijevanger [7], Nevanlinna [13]). Therefore, when looking for simple conditions on T which prevent $\|T^n\|$ from growing linearly with n, in a general Banach space setting, one has to consider conditions (like (1.1), (2.1), (2.3), (2.4)) which are stronger than (2.2).

According to Spijker & Straetemans [21; Theorem 3.1], for $0 < \delta \leq 1/\pi$, condition (2.4) implies

$$\|T^n\| \leq \gamma \quad (n \geq 1). \tag{3.2}$$

Here γ is a constant only depending on K, δ (and not on n, T or X). In view of Theorem 2.3, we conclude that also the conditions (2.1) and (2.3) imply (3.2). This conclusion yields an essential improvement over (3.1) and, in view of Theorem 2.1, to a still further improvement over (1.2).

We remark that the above (indirect) proof of power boundedness under any of the conditions (1.1), (2.1), (2.3), (2.4) does not yield an upper bound for $\|T^n\|$ which depends in a simple way (only) on the constants, like K, L, M, occurring in the resolvent conditions. In connection with this remark, we mention the interesting work of Gibson [6, p.768] and Nevanlinna [13]. These authors presented elegant resolvent conditions which, for an arbitrary given constant γ, are necessary and sufficient in order that $\sup_{n\geq 1} \|T^n\| \leq \gamma$. However, these conditions are not quite easy to check and they are no obvious direct consequence of any of the resolvent conditions discussed in the Sections 1, 2.

We conclude our review by noting that power boundedness of T was proved by Nevanlinna [12; Theorem 4.5.4] under a condition which has similarity to (2.3).

3.2. A direct proof of power boundedness under the Tadmor condition

The following Theorem 3.1 implies power boundedness under Tadmor's condition (2.1). We shall present a direct and short proof of the theorem. Unlike the indirect proof of (3.2) given in Section 3.1, the subsequent proof of Theorem 3.1 yields an upper bound for $\|T^n\|$ which depends in a very transparent way only on L. In view of Section 2, the theorem also implies power boundedness under any of the conditions (1.1), (2.3), (2.4).

Theorem 3.1. *Let T satisfy (2.1). Then*

$$\|T^n\| \leq \frac{e}{2} L^2 \quad (n \geq 1). \tag{3.3}$$

Proof. We have

$$T^n = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \zeta^n (\zeta I - T)^{-1}\, \mathrm{d}\zeta ,$$

where Γ is an arbitrary positively oriented circle $|\zeta| = r$, with $r > 1$. Performing a partial integration, we obtain

$$T^n = \frac{1}{2\pi \mathrm{i}(n+1)} \int_\Gamma \zeta^{n+1} (\zeta I - T)^{-2}\, \mathrm{d}\zeta . \tag{3.4}$$

Applying (2.1) we arrive at

$$\|T^n\| \le \frac{r^{n+1} L^2}{2\pi(n+1)} \cdot J ,$$

where

$$J = \int_{-\pi}^{\pi} \frac{r\, \mathrm{d}t}{|r\mathrm{e}^{\mathrm{i}t} - 1|^2} \tag{3.5a}$$

is the length of the curve $\zeta = (r\mathrm{e}^{\mathrm{i}t} - 1)^{-1}$, $-\pi \le t \le \pi$. Since

$$\left| \frac{1}{r\mathrm{e}^{\mathrm{i}t} - 1} - \frac{1}{r^2 - 1} \right| = \frac{r}{r^2 - 1} ,$$

we see that this curve is the circle with center $1/(r^2-1)$ and radius $r/(r^2-1)$. Hence

$$J = \frac{2\pi r}{r^2 - 1} . \tag{3.5b}$$

Inserting the last expression for J into the upper bound for $\|T^n\|$ obtained above, we arrive at

$$\|T^n\| \le L^2 \cdot F(n, r) ,$$

where

$$F(x, r) = \frac{r^{x+2}}{(x+1)(r^2 - 1)} \quad (\text{for } x \ge 1,\ r > 1) . \tag{3.6a}$$

It can be seen that

$$\min_{r>1} F(x, r) = F(x, \sqrt{1 + 2/x}) \quad \text{and} \quad \sup_{x \ge 1} F(x, \sqrt{1 + 2/x}) = \mathrm{e}/2 . \tag{3.6b}$$

Consequently, our last bound for $\|T^n\|$, with $r = \sqrt{1 + 2/n}$, yields (3.3). □

3.3. Generalizations

Below we shortly discuss generalizations of Theorem 3.1, dealing with the following two resolvent conditions

$$\sigma[T] \subset \{\zeta : |\zeta| \leq 1\}, \quad \text{and}$$
$$\|(\zeta I - T)^{-1}\| \leq \sum_{i=1}^{m} \frac{L_i}{|\zeta - \zeta_i|} \quad (\text{for } 1 < |\zeta| < \rho), \tag{3.7}$$

and

$$\sigma[T] \subset \{\zeta : |\zeta| \leq 1\}, \quad \text{and}$$
$$\|(\zeta I - T)^{-1}\| \leq L\,|\zeta|^{m-1} \prod_{i=1}^{m} \frac{1}{|\zeta - \zeta_i|} \quad (\text{for } 1 < |\zeta| < \rho). \tag{3.8}$$

Here L_i and L are arbitrary constants, m is a positive integer and $1 < \rho \leq \infty$. Further, $\zeta_1, \zeta_2, \cdots, \zeta_m$ denote m different complex values with $|\zeta_i| = 1$ $(1 \leq i \leq m)$. Clearly, if $\zeta_1 = 1$, $m = 1$ and $\rho = \infty$, both of the conditions (3.7) (with $L_1 = L$) and (3.8) reduce to (2.1).

The bound $\|T^n\| \leq \mathrm{e}L^2/2$ of Theorem 3.1 will be generalized to a bound of the form

$$\|T^n\| \leq \alpha \cdot \frac{\mathrm{e}}{2} K^2. \tag{3.9}$$

Both part (a) and part (b) of the subsequent theorem provides us with a generalization of Theorem 3.1.

Theorem 3.2.
(a) *Assume (3.7) and define* $K = \sum_{i=1}^{m} L_i$. *Let* $\nu = 1$ *(for* $\rho \geq \sqrt{\mathrm{e}}$*) and* $\nu = -1 + 1/(\log \rho)$ *(for* $1 < \rho < \sqrt{\mathrm{e}}$*). Then (3.9) holds with* $\alpha = 1$ *(if* $n \geq \nu$*) and* $\alpha = \rho^3(\rho^2 - 1)^{-1}\mathrm{e}^{-1}$ *(if* $1 \leq n < \nu$*).*
(b) *Assume (3.8). Define* $K = L$*, for* $m = 1$*, and*

$$K = L \cdot \sum_{i=1}^{m} \prod_{\substack{k=1 \\ k \neq i}}^{m} |\zeta_i - \zeta_k|^{-1}$$

for $m > 1$. *Then (3.9) holds with* $\alpha = 1$ *(if* $n \geq 1$*).*

In proving Theorem 3.2 we shall make use of the following lemma.

Lemma 3.3. *Let* T, L *and* ζ_i $(1 \leq i \leq m)$ *be given. Suppose (3.8) holds, with some* ρ *satisfying* $1 < \rho < \infty$. *Then (3.8) also holds with* $\rho = \infty$.

Proof of Lemma 3.3. **1.** Let $1 < \rho < \infty$, and assume (3.8). Let $|\zeta| = 1+\epsilon$, $0 < \epsilon < \rho - 1$. Then

$$\frac{1}{2+\epsilon} \leq \frac{1}{\mathrm{d}\,(\zeta, \sigma[T])} \leq \|(\zeta I - T)^{-1}\| \leq \frac{L(1+\epsilon)^{m-1}}{|\zeta - \zeta_1| \cdots |\zeta - \zeta_m|}\,.$$

According to [19, p.353], we have $\max\{\prod_{i=1}^{m} |\zeta - \zeta_i| \,:\, |\zeta| = 1+\epsilon\} \geq 2$. Hence, by letting $\epsilon \to 0$, we arrive at the inequality $L \geq 1$.

2. Define $h(\zeta) = \zeta^{-m+1} \prod_{i=1}^{m} (\zeta - \zeta_i)$ and let $\zeta_0 \in \mathbb{C}$ with $|\zeta_0| \geq \rho$. Let Y denote the normed vector space of all bounded linear operators on the Banach space X. By a well known corollary to the Hahn-Banach theorem (see e.g. [18, Chapter 3]), there is a linear transformation F from Y to $\mathbb{C}$, with

$$F\left(h(\zeta_0)(\zeta_0 I - T)^{-1}\right) = \|h(\zeta_0)(\zeta_0 I - T)^{-1}\|\,, \quad \|F\| = 1\,.$$

The function $f(\zeta) = F\left(h(\zeta)(\zeta I - T)^{-1}\right)$ is holomorphic on $|\zeta| > 1$, with

$$\lim_{\zeta \to \infty} |f(\zeta)| \leq 1 \quad \text{and} \quad |f(\zeta)| \leq L \qquad (\text{for } 1 < |\zeta| < \rho)\,.$$

In view of the maximum modulus theorem, we have $|f(\zeta_0)| \leq \max\{1, L\} = L$. Hence, $\|(\zeta_0 I - T)^{-1}\| \leq L \cdot |h(\zeta_0)|^{-1}$, which proves (3.8), with $\rho = \infty$. □

Proof of Theorem 3.2. **1.** Assume (3.7), and let $n \geq 1$. We choose $r = \min\{\rho, \sqrt{1 + 2/n}\}$. Using (3.4) and writing $x_i = |\zeta - \zeta_i|^{-1}$, we obtain

$$\|T^n\| \leq \frac{r^{n+1}}{2\pi(n+1)} \int_\Gamma \left(\sum_i L_i x_i\right)^2 |\mathrm{d}\zeta| = \frac{r^{n+1}}{2\pi(n+1)} \sum_{i,j} L_i \cdot L_j \cdot \int_\Gamma x_i x_j |\mathrm{d}\zeta|\,.$$

In view of $x_i x_j \leq [(x_i)^2 + (x_j)^2]/2$, there follows

$$\|T^n\| \leq \frac{r^{n+1}}{2\pi(n+1)} \left(\sum_{i=1}^{m} L_i\right)^2 \cdot J\,,$$

where J is as in (3.5a). Applying (3.5b) and using the notation (3.6a), there follows

$$\|T^n\| \leq K^2 \cdot F(n, r)\,. \tag{3.10}$$

First suppose $r = \sqrt{1 + 2/n} \leq \rho$. Then $n \geq 2/(\rho^2 - 1) \geq -1 + 1/\log\rho$, so that $n \geq \nu$. In view of (3.6b) we have $F(n, r) \leq \mathrm{e}/2$, which proves (3.9) with α as stated in part(a).

Next assume $r = \rho \leq \sqrt{1 + 2/n}$, so that $1 \leq n \leq x$, where $x = 2/(\rho^2 - 1)$. It can be seen that

$$F(n, \rho) \leq F(1, \rho) = \frac{\rho^3}{2(\rho^2 - 1)} \quad \text{for } 1 \leq n \leq \nu\,,$$

and

$$F(n, \rho) \leq F(x, \rho) = F(x, \sqrt{1 + 2/x}) \quad \text{for } \nu \leq n \leq x\,.$$

We combine these bounds for $F(n, \rho)$ with (3.10). In view of (3.6b), we obtain (3.9) with α as given in part (a).

2. Assume (3.8). In view of Lemma 3.3 we have (3.8) with $\rho = \infty$. From the partial fraction decomposition

$$\zeta^{m-1} \prod_{k=1}^{m} \frac{1}{\zeta - \zeta_k} = \sum_{i=1}^{m} \frac{\lambda_i \zeta_i^{m-1}}{\zeta - \zeta_i}\,, \quad \text{where } \lambda_i = \prod_{\substack{k=1 \\ k \neq i}}^{m} \frac{1}{\zeta_i - \zeta_k}\,,$$

we see that T satisfies (3.7) with $\rho = \infty$ and $L_i = L\,|\lambda_i \zeta_i^{m-1}|$. Applying the result obtained in part (a), we arrive at the conclusion stated in part (b). □

Remark 3.4. Although Theorem 3.1 is an immediate consequence of Theorem 3.2, we formulated and proved the former theorem separately in view of its simplicity and usefulness, and also because the above proof of Theorem 3.2 is now short thanks to (3.4), (3.5), (3.6).

Remark 3.5. Let any m, T, L_i, ζ_i be given, with $m > 1$ and $|\zeta_i| = 1$ $(1 \leq i \leq m)$. In view of Lemma 3.3, one might conjecture that (3.7) is valid with $\rho = \infty$, whenever the assumption is made that (3.7) holds with some ρ satisfying $1 < \rho < \infty$. By considering the case $T = 0$, it can be seen that this conjecture is false. On the other hand, it can be proved that the assumption just mentioned always implies the resolvent inequality $\|(\zeta I - T)^{-1}\| \leq \sum_{i=1}^{m} L_i' \cdot |\zeta - \zeta_i|^{-1}$ $(1 < |\zeta| < \infty)$, with $L_i' = (\rho+1)(\rho-1)^{-1} L_i$.

4. Relevance of the foregoing to the numerical solution of initial–boundary value problems

Important classes of initial-boundary value problems for linear partial differential equations can be written in the form of an abstract initial value problem

$$v'(t) = A\,v(t) + f(t) \quad (t > 0)\,, \quad v(0) = v_0\,,$$

where v_0, $f(t)$, $v(t)$ belong to an infinite dimensional Banach space X, and A is the infinitesimal generator of an analytic semigroup of linear bounded operators on X (see e.g. [16]). Below we shortly indicate in which way operators T, satisfying the resolvent conditions of Section 2, enter into the numerical solution of such problems.

A standard numerical procedure for solving the above problems consists in replacing them (by means of the finite element or finite difference method) by a semi-discrete problem which can be written in the form

$$u'(t) = A_h u(t) + f_h(t) \quad (t > 0), \quad u(0) = u_0 .$$

Here $h \in (0, h_0]$ is a positive parameter, and u_0, $f_h(t)$, $u(t)$ belong to a finite dimensional Banach space X_h whose dimension tends to infinity when $h \to 0$. Further, A_h is a linear operator from X_h into X_h. In various practical situations (see e.g. [2], [9], [26]) the operator A_h is known to satisfy the following condition:

$$\begin{aligned} &\sigma[A_h] \subset W(\theta), \quad \text{and} \\ &\|(zI - A_h)^{-1}\| \leq \frac{M}{|z|} \quad \text{for all } z \in \mathbb{C} \backslash W(\theta) . \end{aligned} \tag{4.1}$$

In (4.1) we denote by M and θ constants (independent of h) with $M > 0$, $0 < \theta < \pi/2$, and we use the notation $W(\theta) = \{z : z \in \mathbb{C}, |\text{Arg}(-z)| \leq \theta\}$.

Solving the above initial value problem for $u(t)$ numerically by a Runge-Kutta method one arrives at a fully discrete process of the form

$$u_n = T u_{n-1} + r_n \quad (n = 1, 2, \ldots), \tag{4.2}$$

where $T = \phi(\Delta t A_h)$, $\Delta t > 0$ and ϕ is the stability function of the Runge-Kutta method satisfying $\phi(0) = \phi'(0) = 1$. Here $r_n \in X_h$, and $u_n \in X_h$ approximates $u(n\Delta t)$. In the following we assume that (4.1) is fulfilled and that $|\phi(z)| \leq 1$ (for $z \in W(\theta)$).

It can be proved (either directly using arguments similar to those in e.g. [22], or indirectly using Theorem 2.3 and [22, Lemma 3.3], [21, Theorem 2.2]) that the operator $T = \phi(\Delta t A_h)$ satisfies Tadmor's condition (2.1) if $|\phi(\infty)| < 1$ or $\phi(\infty) = 1$. Similarly, in case $|\phi(\infty)| = 1$, $\phi(\infty) \neq 1$, condition (3.7) can be proved (with $\rho = \infty$, $m = 2$). We omit the details of the proofs, but mention that the constants L, L_i in (2.1), (3.7) only depend on ϕ and on the values θ, M in (4.1). The Theorems 3.1, 3.2 show that the numerical process (4.2) is stable in that a constant γ (only depending on ϕ, θ, M) exists such that

$$\|T^n\| \leq \gamma \quad (\text{whenever } 0 < h \leq h_0 , \ \Delta t > 0 , \ n \geq 1) .$$

We refer to [1], [14], [15] for stability results relevant to the situation (4.1), derived in a different way.

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