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The Stability of the θ-methods in the Numerical Solution of Delay Differential Equations

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This paper deals with the stability analysis of numerical methods for the solution of delay differential equations. We focus on the behaviour of the one-leg θ-method and the linear θ-method in the solution of the linear test equation $U'(t) = \lambda U(t) + \mu U(t-\tau)$, with $\tau > 0$ and complex λ, μ. The stability regions for both of these methods are determined. The regions turn out to be equal to each other only if $\theta = 0$ or $\theta = 1$.

1. Introduction

1.1 *The θ-methods*

THIS PAPER deals with the numerical solution of the initial-value problem

$$U'(t) = f\big(t, U(t), U(\alpha[t])\big) \quad (t \geq 0), \qquad U(t) = U_0(t) \quad (t \leq 0). \tag{1.1}$$

Here f, U_0, α denote given functions with $\alpha[t] \leq t$ whereas $U(t)$ is unknown (for $t > 0$). With the so-called θ-methods, one can compute approximations u_n to $U(t)$ at the *gridpoints* $t = nh$, where $h > 0$ denotes the *stepsize* and $n = 1, 2, 3, \ldots$.

The following method was considered, for example, in [14, 15, 16]

$$u_{n+1} = u_n + hf\big((n+\theta)h, u^h\big((n+\theta)h\big), u^h(\alpha[(n+\theta)h])\big) \qquad (n = 0, 1, 2, \ldots). \tag{1.2a}$$

Here θ is a parameter, with $0 \leq \theta \leq 1$, specifying the method. Further $u_0 = U_0(0)$, $u^h(t) = U_0(t)$ (for $t \leq 0$), and $u^h(t)$ with $t > 0$ is defined by piecewise linear interpolation, i.e.

$$u^h(t) = \frac{t - kh}{h} u_{k+1} + \frac{(k+1)h - t}{h} u_k \quad \text{for } kh < t \leq (k+1)h;\ k = 0, 1, 2, \ldots.$$

In addition to (1.2a), we consider the method

$$u_{n+1} = u_n + h\big[\theta \cdot f\big((n+1)h, u_{n+1}, u^h(\alpha[(n+1)h])\big) + (1-\theta) \cdot f\big(nh, u_n, u^h(\alpha[nh])\big)\big] \quad (n = 0, 1, 2, \ldots), \tag{1.2b}$$

which was considered, for example, in [4, 9, 14].

In discussing the above two methods it is instructive to consider the processes to which (1.2a), (1.2b) reduce in the case where the delay term $U(\alpha[t])$ in (1.1) is absent. In fact, if (1.1) were to stand for an initial-value problem

$$U'(t) = f(t, U(t)) \quad (t \geq 0), \qquad U(0) = u_0,$$

then (1.2a) and (1.2b) would reduce to

$$u_{n+1} = u_n + hf((n+\theta)h, \theta u_{n+1} + (1-\theta)u_n) \quad (n \geq 0), \tag{1.3a}$$

$$u_{n+1} = u_n + h[\theta f((n+1)h, u_{n+1}) + (1-\theta)f(nh, u_n)] \quad (n \geq 0), \tag{1.3b}$$

respectively. Method (1.3a) can be viewed as a one-leg multistep method, and (1.3b) as a linear multistep method (cf. [6]). Accordingly, we shall refer to (1.2a) and (1.2b) by the terms *one-leg θ-method* and *linear θ-method* respectively.

In spite of their simplicity the above methods have often been used successfully in actual calculations (even the well-known Crank–Nicholson method for the numerical solution of parabolic differential equations can be viewed as a θ-method, with $\theta = \frac{1}{2}$, applied to some large system of ordinary differential equations).

The stability behaviour of the methods (1.3a,b) has been studied extensively (see e.g. [3, 6, 11, 13]). However, the methods (1.2a,b) can exhibit instability phenomena deviating considerably and unexpectedly from what is known to hold for (1.3a,b). This paper aims at contributing to more insight into these phenomena.

1.2 *The test problem*

Following [1, 2, 4, 5, 9, 14, 15, 16], we shall assess the stability of the methods (1.2a) and (1.2b) by analysing their stability behaviour in the numerical solution of the test problem

$$U'(t) = \lambda U(t) + \mu U(t-\tau) \quad (t \geq 0), \qquad U(t) = U_0(t) \quad (t \leq 0). \tag{1.4}$$

Here $\tau > 0$ is a constant delay, and λ, μ denote complex constants whereas U_0 is any given, continuous, complex-valued function. From [2] it follows that the solution $U(t)$ to (1.4) tends to zero as $t \to \infty$, for all U_0, if

$$\text{Re}(\lambda) < -|\mu|. \tag{1.5}$$

Following, for example, [1, 2, 4, 5, 8, 9, 14, 15, 16], we shall concentrate in the following on the question of how far this stability property of (1.4) is carried over to the corresponding numerical processes.

Applying (1.2a,b) to (1.4) we arrive at the recurrence relations

$$(1-\theta x)u_{n+1} = [1 + (1-\theta)x]u_n + \sigma y u_{n-r+1} + (1-\sigma) y u_{n-r} \quad (n \geq m), \tag{1.6a}$$

$$\begin{aligned}(1-\theta x)u_{n+1} = [1 + (1-\theta)x]u_n &+ \theta y[\delta u_{n-m+2} + (1-\delta)u_{n-m+1}] \\ &+ (1-\theta)y[\delta u_{n-m+1} + (1-\delta)u_{n-m}] \quad (n \geq m),\end{aligned} \tag{1.6b}$$

respectively. Here m is the smallest integer with $\tau h^{-1} \leq m$ and

$$\begin{aligned} &x = h\lambda, \qquad y = h\mu, \qquad \delta = m - \tau h^{-1}, \\ &\sigma = \theta + \delta \quad \text{and} \quad r = m \quad \text{if } 0 \leq \delta < 1-\theta, \\ &\sigma = \theta + \delta - 1 \quad \text{and} \quad r = m-1 \quad \text{if } 1-\theta \leq \delta < 1. \end{aligned} \tag{1.7}$$

The values u_k $(k = 0, 1, \ldots, m)$, which are needed to start the numerical processes (1.6a,b), are obtained from the given function U_0 and from applications of (1.2a,b) (with $n = 0, 1, \ldots, m-1$).

We note that if $\alpha[t]$ in (1.1) equals $t-\tau$ with fixed delay $\tau > 0$, a natural choice for h in (1.2) seems to be $h = \tau m^{-1}$, yielding $\delta = 0$ in (1.7). However, the uniform use of one such constant stepsize h can be rather inefficient. Moreover, in the general case of variable delay $t - \alpha[t]$ no such natural stepsize h exists. Therefore, it is important to include in the stability analysis, based on (1.4), the case of general stepsizes $h = \tau(m-\delta)^{-1}$ with arbitrary integer $m \geqslant 1$ and arbitrary $\delta \in [0, 1)$.

We now give our basic definitions for given $\delta \in [0, 1)$ and $x, y \in \mathbb{C}$. Method (1.2a) is called *δ-stable at* (x, y) if and only if any application of (1.2a) to problem (1.4) generates numerical approximations u_n that tend to zero (as $n \to \infty$) whenever λ, μ, τ, and h satisfy $\tau > 0$, $h > 0$, $h\lambda = x$, $h\mu = y$, and $m - \tau h^{-1} = \delta$. Similarly, method (1.2b) is called δ-stable at (x, y) if the above holds for applications of (1.2b) to (1.4). It can be seen that (1.2a) is δ-stable at (x, y) if and only if, for all integers $m \geqslant 1$ and all starting values $u_0, u_1, \ldots, u_m \in \mathbb{C}$, the relations (1.6a) imply that $\lim_{n\to\infty} u_n = 0$. An analogous remark applies to (1.2b) and (1.6b). We define method (1.2a) or (1.2b) to be *stable at* (x, y) if and only if it is δ-stable for all $\delta \in [0, 1)$.

1.3 *Results on stability regions to be found in the literature*

The subset of $\mathbb{C}^2$ consisting of all pairs (x, y) at which a method is stable or δ-stable is called its *stability region* or *δ-stability region* respectively. For the one-leg θ-method we denote these sets by S_θ and $S_{\theta,\delta}$, and for the linear θ-method by $\tilde{S}_\theta$ and $\tilde{S}_{\theta,\delta}$ respectively. Clearly

$$S_\theta = \bigcap_{0 \leqslant \delta < 1} S_{\theta,\delta}, \qquad \tilde{S}_\theta = \bigcap_{0 \leqslant \delta < 1} \tilde{S}_{\theta,\delta}.$$

In order to visualize the above sets it is convenient to introduce, for given complex y, the subsets of the complex plane

$$R_\theta(y, \delta) = \{x : (x, y) \in S_{\theta,\delta}\}, \qquad R_\theta(y) = \{x : (x, y) \in S_\theta\}, \tag{1.8a}$$

$$\tilde{R}_\theta(y, \delta) = \{x : (x, y) \in \tilde{S}_{\theta,\delta}\}, \qquad \tilde{R}_\theta(y) = \{x : (x, y) \in \tilde{S}_\theta\}. \tag{1.8b}$$

From (1.6) it is evident that these four sets coincide if $y = 0$. This can be rephrased by saying that the two methods (1.3a,b) exhibit just the same stability behaviour when applied to the test equation $U'(t) = \lambda U(t)$. We shall see that this does not remain true for the two methods (1.2a,b) applied to the equation $U'(t) = \lambda U(t) + \mu U(t-\tau)$ with $\mu \neq 0$.

In view of condition (1.5) we introduce the set H consisting of all pairs $(x, y) \in \mathbb{C}^2$ with

$$\operatorname{Re} x < -|y|.$$

Many authors have dealt with the important question of whether H is contained in the (δ-)stability regions under consideration.

In [16] Zennaro obtained interesting stability results for general Runge–Kutta methods. Specializing his results to the situation at hand, it is easy to determine explicitly the set $S_{\theta,0} = \bar{S}_{\theta,0}$ for $0 \leq \theta \leq 1$, and to show that $H \subset S_{\theta,0}$ for $\frac{1}{2} \leq \theta \leq 1$, but $H \not\subset S_{\theta,0}$ for $0 \leq \theta < \frac{1}{2}$.

Stability results for arbitrary $\delta \in [0, 1)$ were obtained by Watanabe & Roth [14], and by Calvo & Grande [4]. These authors showed that

$$H \subset \bar{S}_\theta \quad \text{for } \tfrac{1}{2} \leq \theta \leq 1.$$

On the other hand, Watanabe & Roth [14] and Zennaro [15] showed

$$H \not\subset S_\theta \quad \text{for } \theta = \tfrac{1}{2}.$$

Zennaro [16: p. 316] formulated a conjecture implying that the sets S_θ and $\bar{S}_\theta$ can be expected to be much smaller than $S_{\theta,0}$ and $\bar{S}_{\theta,0}$ respectively.

Following [4] we introduce, for $0 \leq \theta < \frac{1}{2}$, the set H_θ consisting of all pairs $(x, y) \in \mathbb{C}^2$ with

$$|x + (1-2\theta)^{-1}| + |y| < (1-2\theta)^{-1}.$$

Note that $H_\theta \subset H_{\theta'} \subset H$ for $0 \leq \theta \leq \theta' < 1/2$.

Calvo and Grande [4] proved that

$$H_\theta \subset \bar{S}_\theta \quad \text{for } 0 \leq \theta < \tfrac{1}{2},$$

and they conjectured that $H_\theta = \bar{S}_\theta \cap H$.

1.4 *Scope of this paper*

The main purpose of the present paper is to determine all sets (1.8) and to settle the two conjectures mentioned above.

In Section 2 we derive a general stability theorem by using, similar to that in [14], the principle of the argument.

In Section 3 we determine the sets (1.8a) by applying the stability theorem from Section 2. Similarly we determine in Section 4 the sets (1.8b).

In the final Section 5 it becomes clear that Zennaro's conjecture is partially true in that S_θ is smaller than $S_{\theta,0}$ for $0 < \theta < 1$. However, $S_\theta = S_{\theta,0}$ for $\theta = 0$ or $\theta = 1$, and $\bar{S}_\theta = \bar{S}_{\theta,0}$ for $0 \leq \theta \leq 1$.

Further, it is shown that an infinite number of pairs $(x, y) \in \bar{S}_\theta \cap H$ exist with $(x, y) \notin H_\theta$ (for $\theta = 0$). This disproves the conjecture of [4].

Section 5 also presents a comparison, for all $\theta \in [0, 1]$, of the sets S_θ and $\bar{S}_\theta$, and the question is settled for which θ these sets contain H. The paper concludes with some numerical illustrations.

2 General stability considerations

To the recurrence relations (1.6a,b) we adjoin the characteristic polynomials

$$P_m(z, \delta) = z^{m+1} - [1 + (1-\theta)x](1-\theta x)^{-1} z^m - p(z, \delta), \tag{2.1a}$$

$$\bar{P}_m(z, \delta) = z^{m+1} - [1 + (1-\theta)x](1-\theta x)^{-1} z^m - \bar{p}(z, \delta), \tag{2.1b}$$

respectively with

$$p(z,\delta)=\frac{[\sigma z^{q+1}+(1-\sigma)z^q]y}{1-\theta x},\qquad q=\begin{cases}0 & \text{if } \delta\in[0,1-\theta),\\ 1 & \text{if } \delta\in[1-\theta,1),\end{cases} \tag{2.2a}$$

$$\tilde{p}(z,\delta)=\frac{(\delta z+1-\delta)(\theta z+1-\theta)y}{1-\theta x}. \tag{2.2b}$$

We recall that a polynomial, all of whose zeros lie within the open unit disk in the complex plane, is called a *Schur polynomial* (cf. e.g. [4], [7], [10]). By a well-known property of Schur polynomials (see e.g. [7], [10]) we obtain the following lemma.

LEMMA 2.1 (a) *Method* (1.2*a*) *is* δ-*stable at* (x, y) *if and only if* $P_m(z,\delta)$ *is a Schur polynomial for all* $m \geq 1$.

(b) *Method* (1.2*b*) *is* δ-*stable at* (x, y) *if and only if* $\tilde{P}_m(z,\delta)$ *is a Schur polynomial for all* $m \geq 1$.

In view of this lemma and the structure of the above polynomials $P_m(z,\delta)$ and $\tilde{P}_m(z,\delta)$, we shall derive in the rest of this section a condition that is necessary and sufficient in order that

$$P_m(z)=z^{m+1}-cz^m-p(z) \tag{2.3}$$

is a Schur polynomial for all $m \geq \max(1, k-1)$. Here c is any given complex constant and $p(z)$ is an arbitrary fixed polynomial of a degree $k \geq 0$.

In all of the following, C will stand for the (positively oriented) unit circle in the complex plane. If D is any arc of C and $f(z)\neq 0$ for $z\in D$, then $\Delta[\arg f(z); D]$ will denote the increment of the argument of $f(z)$ when z runs through D.

LEMMA 2.2 *Let* $P_m(z)$, *given by* (2.3), *be a Schur polynomial for all* $m \geq \max(1, k-1)$. *Then* $|p(z)| \leq |z-c|$ for all $z \in C$.

Proof. Using the fact that the number of zeros of a polynomial does not exceed its degree, it can be seen that either $|p(z)|=|z-c|$ for all z with $|z|=1$, or the number of points z with $|z|=1$, $|p(z)|=|z-c|$ does not exceed $\max(2k, 2)$. Therefore C can be split up into (maximal) arcs $D_1, D_2, \ldots, D_q, E_1, E_2, \ldots, E_r$ such that

$$|p(z)|\leq|z-c| \quad \text{for all } z\in D_j,\ 1\leq j\leq q;\quad 0\leq q\leq \max(2k,2),$$
$$|p(z)|>|z-c| \quad \text{for all } z\in E_j,\ 1\leq j\leq r;\quad 0\leq r\leq \max(2k,2).$$

If the point c lies on any of the arcs D_j, then $p(c)=0$ and therefore also $P_m(c)=0$. The latter equality would contradict the fact that $P_m(z)$ is a Schur polynomial. Hence

$$c \notin D_j. \tag{2.4}$$

It follows that for $z \in D_j$, we can write $P_m(z)$ in the form

$$P_m(z)=z^m(z-c)[1-\delta(z)],$$

where $\delta(z) = (z^{m+1} - cz^m)^{-1}p(z)$ has a modulus $|\delta(z)| \leq 1$ and $\delta(z) \neq 1$. Using the equality

$$\Delta[\arg P_m(z); D_j] = \Delta[\arg z^m; D_j] + \Delta[\arg(z-c); D_j] + \Delta[\arg(1-\delta(z)); D_j] \quad (2.5)$$

and denoting the length of the arc D_j by $|D_j|$, we arrive at the inequality

$$\Delta[\arg P_m(z); D_j] \leq m\,|D_j| + 3\pi \quad (j = 1, 2, \ldots, q). \quad (2.6)$$

Similarly we have, for $z \in E_j$, the representation

$$P_m(z) = p(z)(-1 + \varepsilon(z)),$$

where $\varepsilon(z) = (z^{m+1} - cz^m)p(z)^{-1}$ has a modulus $|\varepsilon(z)| < 1$. Using $\Delta[\arg P_m(z); E_j] = \Delta[p(z); E_j] + \Delta[-1 + \varepsilon(z); E_j]$ we thus arrive at the inequality

$$\Delta[\arg P_m(z); E_j] \leq \Delta[p(z); E_j] + \pi \quad (j = 1, 2, \ldots, r). \quad (2.7)$$

Since $P_m(z)$ is a Schur polynomial, we obtain, by an application of the principle of the argument, the equality $(m+1)2\pi = \Delta[\arg P_m(z); C]$. In view of (2.6) and (2.7) we thus have, for $m = 1, 2, 3, \ldots$,

$$(m+1)2\pi \leq \sum_{j=1}^{q} (m\,|D_j| + 3\pi) + \sum_{j=1}^{r} (\Delta[p(z); E_j] + \pi).$$

Dividing both members of this inequality by $(m+1)$ and letting $m \to \infty$, it follows that

$$2\pi \leq |D_1| + |D_2| + \cdots + |D_q|.$$

Since the lengths of all arcs D_j and E_j add to 2π, we see that the number r of arcs E_j equals $r = 0$. □

LEMMA 2.3 *Let $P_m(z)$, given by (2.3), be a Schur polynomial for all $m \geq \max(1, k-1)$. Then $|c| < 1$.*

Proof. Using the same notation as in the above proof, we deduce from Lemma 2.2 that $q = 1$, $r = 0$, and $C = D_1$.

An application of (2.4) with $j = 1$ shows that $|c| \neq 1$.

Applying (2.5) with $j = 1$ we obtain

$$(m+1)2\pi = \Delta[\arg P_m(z); D_1] = m2\pi + \Delta[\arg(z-c); D_1].$$

Hence

$$\Delta[\arg(z-c); D_1] = 2\pi,$$

which implies $|c| < 1$. □

Using the above two lemmas we can prove the main result of this section.

THEOREM 2.4 *$P_m(z)$, given by (2.3), is a Schur polynomial for all $m \geq \max(1, k-1)$, if and only if the following three conditions are fulfilled:*

$$|p(z)| \leq |z - c| \quad \textit{for all } z \in C, \quad (2.8)$$

$$|c| < 1, \quad (2.9)$$

$$P_m(z) \neq 0 \quad \textit{for all } m \geq \max(1, k-1),\ z \in C. \quad (2.10)$$

Proof. (a) Assume $P_m(z)$ is a Schur polynomial for all $m \geq \max(1, k-1)$. Then (2.8) and (2.9) follow from Lemmas 2.2 and 2.3 respectively, whereas (2.10) is trivially fulfilled.

(b) Assume (2.8), (2.9), (2.10), and let $m \geq \max(1, k-1)$. It can be seen that $P_m(z)$ is a Schur polynomial by an application of Rouché's theorem (cf. [12]), or more directly by applying the arguments already used in the proof of Lemma 2.2. In the latter case one only has to apply (2.5) with $j = 1$ and $D_1 = C$ so as to get

$$\Delta[\arg P_m(z); C] = (m+1)2\pi,$$

and to apply the principle of the argument. □

3. Stability regions of the one-leg θ-methods

3.1 *An application of Theorem* 2.4

In this section we will determine the sets $R_\theta(y, \delta)$ and $R_\theta(y)$ defined in Section 1.3. We shall use the notation (1.7), (2.1a), (2.2a), and

$$\varphi(x) = [1 + (1-\theta)x](1-\theta x)^{-1}. \tag{3.1}$$

In view of Lemma 2.1a we see, by an application of Theorem 2.4 with $P_m(z) = P_m(z, \delta)$, that $x \in R_\theta(y, \delta)$ if and only if

$$P_m(z, \delta) \neq 0 \quad \text{for all } z \in C \text{ and } m \geq 1, \tag{3.2}$$

$$|\varphi(x)| < 1 \quad \text{and} \quad |p(z, \delta)| \leq |z - \varphi(x)| \quad \text{for all } z \in C. \tag{3.3}$$

Since

$$|\varphi(x)| < 1 \quad \text{and} \quad |p(z, \delta)| < |z - \varphi(x)| \quad \text{for all } z \in C \tag{3.4}$$

implies the properties (3.2) and (3.3), it follows that (3.4) is a condition which is sufficient in order that $x \in R_\theta(y, \delta)$.

3.2 *Determination of the set* $R_\theta(y, \delta)$

When dealing with the above condition $|\varphi(x)| < 1$, it is convenient to use the notation

$$\rho = (1-2\theta)^{-1} \quad \text{for } \theta \neq \tfrac{1}{2},$$

and to introduce the generalized disk D, defined by

$$D = \{x : |x + \rho| < \rho\} \quad \text{if } 0 \leq \theta < \tfrac{1}{2},$$
$$D = \{x : \operatorname{Re} x < 0\} \quad \text{if } \theta = \tfrac{1}{2},$$
$$D = \{x : |x + \rho| > -\rho\} \quad \text{if } \tfrac{1}{2} < \theta \leq 1.$$

With $\partial(D)$ we shall denote the collection of (finite) boundary points of D.

LEMMA 3.1 (a) *$x \in D$ if and only if $|\varphi(x)| < 1$.*
(b) *$\xi \in \partial(D)$ if and only if there is a $z \in C$ with $1 - \theta + \theta z \neq 0$ and $\xi = (z-1)(1-\theta+\theta z)^{-1}$.*

Proof. Part (a) follows from a straightforward calculation. Part (b) is easily proved by noticing that

$$1-\theta+\theta z\neq 0,\qquad x=(z-1)(1-\theta+\theta z)^{-1}$$

is equivalent to

$$1-\theta x\neq 0,\qquad z=\varphi(x).\qquad\square$$

The subsequent lemma concerns the last inequalities occurring in the above conditions (3.3) and (3.4).

LEMMA 3.2 *Let x be any given complex number with $1-\theta x\neq 0$. Then the condition*

$$|p(z,\delta)|<|z-\varphi(x)|\quad \textit{for all } z\in C$$

is equivalent to the requirement that

$$|[1+(\sigma-\theta)\xi]y|<|x-\xi|\quad \textit{for all } \xi\in\partial(D), \tag{3.5a}$$
$$|(\sigma-\theta)y|<1\qquad \textit{if } \theta=\tfrac{1}{2}. \tag{3.5b}$$

Moreover, this equivalence remains true if in the above three inequalities the sign $<$ is replaced simultaneously by $\leq$.

Proof. From (2.2a) and (3.1) we see that $|p(z,\delta)|<|z-\varphi(x)|$ is equivalent to

$$|(\sigma z+1-\sigma)y|<|(1-\theta+\theta z)x-(z-1)|\,.$$

By Lemma 3.1b it follows that the latter inequality holds for all $z\in C$ with $1-\theta+\theta z\neq 0$ if and only if (3.5a) is fulfilled. If $z\in C$ with $1-\theta+\theta z=0$, then $\theta=\frac{1}{2}$, $z=-1$, and the inequality reduces to (3.5b).

The case where $\leq$ replaces $<$ is proved in a similar fashion. $\square$

Along with (3.5) we also deal with the requirement

$$|[1+(\sigma-\theta)\xi]y|\leq|x-\xi|\quad \text{for all } \xi\in\partial(D), \tag{3.6a}$$
$$|(\sigma-\theta)y|\leq 1\qquad \text{if } \theta=\tfrac{1}{2}. \tag{3.6b}$$

Denoting the sets of all x satisfying (3.4) or (3.3) by $A(y,\delta)$ and $B(y,\delta)$ respectively, we have, by Lemmas 3.1a and 3.2,

$$A(y,\delta)=\{x: x\in D \text{ and (3.5) holds}\}. \tag{3.7}$$

In view of the above lemmas we also can conclude that

$$B(y,\delta)=\{x: x\in D \text{ and (3.6) holds}\}. \tag{3.8}$$

In view of Section 3.1 we thus have the following theorem.

THEOREM 3.3 *Let $0\leq\theta\leq 1$, $0\leq\delta<1$, and y be given. Let $A(y,\delta)$ and $B(y,\delta)$ be as in* (3.7) *and* (3.8). *Then $R_\theta(y,\delta)$ equals the set of all $x\in B(y,\delta)$ satisfying* (3.2). *Further*

$$A(y,\delta)\subset R_\theta(y,\delta)\subset B(y,\delta).$$

In Section 5.3 we shall see that in general $A(y,\delta)\neq R_\theta(y,\delta)\neq B(y,\delta)$.

Applying this theorem with $\delta=0$ we easily obtain the following.

COROLLARY 3.4 *Let* $0 \leq \theta \leq 1$ *and* y *be given. Let* $\rho = (1-2\theta)^{-1}$ *if* $\theta \neq \frac{1}{2}$. *Then*

$$A(y, 0) \subset R_\theta(y, 0) \subset B(y, 0),$$

with

$$A(y, 0) = \{x : |x+\rho| < \rho - |y|\}, \quad \text{if } 0 \leq \theta < \tfrac{1}{2},$$
$$A(y, 0) = \{x : \operatorname{Re} x < -|y|\}, \quad \text{if } \theta = \tfrac{1}{2},$$
$$A(y, 0) = \{x : |x+\rho| > -\rho + |y|\}, \quad \text{if } \tfrac{1}{2} < \theta \leq 1,$$

and $B(y, 0)$ *being equal to the closure of* $A(y, 0)$.

We note that the above characterization of $R_\theta(y, 0)$ can also be deduced from the very general results in [16]. But the above derivation of the corollary, which is based on Theorem 2.4, is essentially different from (and shorter than) the proof via the material in [16].

3.3 *Determination of* $R_\theta(y)$

In combination with Theorem 3.3 the following theorem provides a complete characterization of the set $R_\theta(y)$. We define the sets $A(y, 1)$ and $R_\theta(y, 1)$ by

$$A(y, 1) = A(y, 0), \qquad R_\theta(y, 1) = R_\theta(y, 0).$$

THEOREM 3.5 *Let* $0 \leq \theta \leq 1$, $0 \leq \delta < 1$, *and* y *be given. Then*

$$A(y, 1-\theta) \subset A(y, \delta), \qquad R_\theta(y, 1-\theta) \subset R_\theta(y, \delta),$$

and therefore

$$R_\theta(y) = R_\theta(y, 1-\theta).$$

Proof. (a) We use the definitions $P_m(z, 1) = P_m(z, 0)$ and $p(z, 1) = p(z, 0)$. Assume, with no loss of generality, that $\delta \neq 1-\theta$ when $0 < \theta \leq 1$, $\delta \neq 0$ when $\theta = 0$. Since $0 < \sigma < 1$ (cf. (1.7)) we obtain from (2.2a)

$$|p(z, \delta)| = |\sigma z + (1-\sigma)|\,|p(z, 1-\theta)|, \qquad |\sigma z + (1-\sigma)| < 1 \quad \text{for } 1 \neq z \in C, \tag{3.9a}$$

$$p(z, \delta) = p(z, 1-\theta) \quad \text{for } z = 1. \tag{3.9b}$$

(b) Let $x \in A(y, 1-\theta)$. By the definition of $A(y, 1-\theta)$ we have

$$|\varphi(x)| < 1, \qquad |p(z, 1-\theta)| < |z - \varphi(x)| \quad \text{for } z \in C.$$

Applying (3.9) there follows $|p(z, \delta)| < |z - \varphi(x)|$ for $z \in C$. By the definition of $A(y, \delta)$ we thus have $x \in A(y, \delta)$. Hence $A(y, 1-\theta) \subset A(y, \delta)$.

(c) Let $x \in R_\theta(y, 1-\theta)$. Similarly, as above, we obtain here

$$|\varphi(x)| < 1, \qquad |p(z, \delta)| \leq |p(z, 1-\theta)| \leq |z - \varphi(x)| \quad \text{for } z \in C.$$

By the definition of $B(y, \delta)$ we thus have $x \in B(y, \delta)$.

Suppose (3.2) is violated. Then there is a $z^* \in C$ with $P_m(z^*, \delta) = 0$. By (2.1a) and (3.9) there follows

$$\begin{aligned} |z^* - \varphi(x)| = |p(z^*, \delta)| &= |z^*\sigma + (1-\sigma)|\,|p(z^*, 1-\theta)| \\ &\leq |z^*\sigma + (1-\sigma)|\,|z^* - \varphi(x)|\,. \end{aligned}$$

Consequently $z^* = 1$. This is a contradiction since $P_m(1, \delta) = P_m(1, 1-\theta) \neq 0$.
It follows that (3.2) holds, and therefore $R_\theta(y, 1-\theta) \subset R_\theta(y, \delta)$.
(d) The proof is completed by noting that

$$R_\theta(y) = \bigcap_{0 \leqslant \delta < 1} R_\theta(y, \delta). \qquad \square$$

4. Stability regions of the linear θ-methods

4.1 *An application of Theorem* 2.4

The sets $\bar{R}_\theta(y, \delta)$ and $\bar{R}_\theta(y)$ defined in Section 1.3 can be determined using arguments analogous to those used in Section 3. We still use the notations (1.7), (2.1b), (2.2b), and (3.1). Similarly, as in Section 3.1, we arrive here at the conclusion that $x \in \bar{R}_\theta(y, \delta)$ if and only if

$$\bar{P}_m(z, \delta) \neq 0 \quad \text{for all } z \in C \text{ and } m \geqslant 1, \tag{4.1}$$

$$|\varphi(x)| < 1 \quad \text{and} \quad |\bar{p}(z, \delta)| \leqslant |z - \varphi(x)| \quad \text{for all } z \in C. \tag{4.2}$$

Moreover, the requirement

$$|\varphi(x)| < 1 \quad \text{and} \quad |\bar{p}(z, \delta)| < |z - \varphi(x)| \quad \text{for all } z \in C \tag{4.3}$$

is sufficient in order that $x \in \bar{R}_\theta(y, \delta)$.

4.2 *Determination of the set* $\bar{R}_\theta(y, \delta)$

We use the notations ρ and D introduced in the beginning of Section 3.2. Analogously to Lemma 3.2 we have here the following lemma.

LEMMA 4.1 *Let x be any given complex number with $1 - \theta x \neq 0$. Then the condition*

$$|\bar{p}(z, \delta)| < |z - \varphi(x)| \quad \textit{for all } z \in C$$

is equivalent to

$$|1 + \delta\xi(1-\theta\xi)^{-1}|\,|y| < |x - \xi| \quad \textit{for all } \xi \in \partial(D). \tag{4.4}$$

Moreover, this equivalence remains true if, in the above two inequalities, the sign $<$ is replaced simultaneously by $\leqslant$.

Proof. From (2.2b) and (3.1) we see that the first inequality in the lemma is equivalent to

$$|(1 - \delta + \delta z)(1 - \theta + \theta z)y| < |(1 - \theta + \theta z)x - (z - 1)|.$$

Using Lemma 3.1(b), it follows that the latter inequality is valid for all $z \in C$ if and only if (4.4) holds. The case where $\leqslant$ replaces $<$ is proved similarly. $\square$

In addition to (4.4), we deal with the requirement

$$|1 + \delta\xi(1-\theta\xi)^{-1}|\,|y| \leqslant |x - \xi| \quad \text{for all } \xi \in \partial(D). \tag{4.5}$$

Denoting the sets of all x satisfying (4.3) or (4.2) by $\tilde{A}(y, \delta)$ and $\tilde{B}(y, \delta)$ respectively, we have, in view of Lemmas 3.1(a) and 4.1,

$$\tilde{A}(y, \delta) = \{x : x \in D \text{ and (4.4) holds}\}, \tag{4.6}$$

$$\tilde{B}(y, \delta) = \{x : x \in D \text{ and (4.5) holds}\}. \tag{4.7}$$

In view of Section 4.1 we arrive at the following.

THEOREM 4.2 *Let* $0 \leq \theta \leq 1$, $0 \leq \delta < 1$, *and* y *be given. Let* $\tilde{A}(y, \delta)$ *and* $\tilde{B}(y, \delta)$ *be as in* (4.6) *and* (4.7). *Then* $\tilde{R}(y, \delta)$ *equals the set of all* $x \in \tilde{B}(y, \delta)$ *satisfying* (4.1). *Further*

$$\tilde{A}(y, \delta) \subset \tilde{R}_\theta(y, \delta) \subset \tilde{B}(y, \delta).$$

4.3 *Determination of* $\tilde{R}_\theta(y)$

In combination with the above, the following theorem provides a complete characterization of the set $\tilde{R}_\theta(y)$.

THEOREM 4.3 *Let* $0 \leq \theta \leq 1$, $0 \leq \delta < 1$, *and* y *be given. Then*

$$\tilde{A}(y, 0) = A(y, 0) \subset \tilde{A}(y, \delta), \qquad \tilde{R}_\theta(y, 0) = R_\theta(y, 0) \subset \tilde{R}_\theta(y, \delta),$$

and therefore

$$\tilde{R}_\theta(y) = \tilde{R}_\theta(y, 0).$$

Proof. Assume $\delta \neq 0$. From (2.2b) it follows that

$$|\tilde{p}(z, \delta)| = |\delta z + 1 - \delta|\, |\tilde{p}(z, 0)|, \qquad |\delta z + 1 - \delta| < 1 \quad \text{for } 1 \neq z \in C, \tag{4.8a}$$

$$\tilde{p}(z, \delta) = \tilde{p}(z, 0) \quad \text{for } z = 1. \tag{4.8b}$$

Using (4.8a), similarly as (3.9a) was used in the second part of the proof of Theorem 3.5, it follows that $\tilde{A}(y, 0) \subset \tilde{A}(y, \delta)$. By using (4.8a,b), similarly as (3.9a,b) were used in the third part of the proof of Theorem 3.5, it follows that $\tilde{R}_\theta(y, 0) \subset \tilde{R}_\theta(y, \delta)$.

From (2.1) and (2.2) we see that $\tilde{P}_m(z, 0) = P_m(z, 0)$. Consequently,

$$\tilde{A}(y, 0) = A(y, 0), \qquad \tilde{R}_\theta(y, 0) = R_\theta(y, 0).$$

The proof is completed by using the definition of $\tilde{R}_\theta(y)$. □

5. Concluding remarks and numerical experiments

5.1 *Comparing the sets* S_θ *and* $\tilde{S}_\theta$

From Theorem 3.5 it follows that $S_{\theta,\delta} \subset S_{\theta,0}$ (for $0 < \delta = 1 - \theta < 1$). Applying Theorem 3.3, with $\delta = 0$ and with $\delta = 1 - \theta$, it can easily be verified that $S_{\theta,0}$ contains pairs (x, y) that do not belong to $S_{\theta,\delta}$ (for $0 < \delta = 1 - \theta < 1$). Consequently $S_{\theta,\delta} \subsetneq S_{\theta,0}$ (for $0 < \delta = 1 - \theta < 1$), and therefore

$$S_\theta \subsetneq S_{\theta,0} \quad \text{for } 0 < \theta < 1. \tag{5.1}$$

By Theorem 3.5 (with $\theta = 0$ and $\theta = 1$) we also have

$$S_\theta = S_{\theta,0} \quad \text{for } \theta = 0,\ 1. \tag{5.2}$$

From Theorem 4.3 it follows that

$$\tilde{S}_\theta = \tilde{S}_{\theta,0} \quad \text{for } 0 \leqslant \theta \leqslant 1, \tag{5.3}$$

which together with (5.2) shows that Zennaro's conjecture (cf. Section 1.3) is only partially true.

In view of (5.3) and (5.1), we see that

$$\tilde{S}_\theta = \tilde{S}_{\theta,0} = S_{\theta,0} \supsetneqq S_\theta \quad \text{for } 0 < \theta < 1.$$

We thus have the following theorem.

THEOREM 5.1 *For all $\theta \in (0\,,1)$ the linear θ-method* (1.2*b*) *has a larger stability region than the corresponding one-leg θ-method* (1.2*a*). *Only for $\theta = 0$ and $\theta = 1$ has the linear method the same stability region as the one-leg θ-method.*

5.2 *Properties of the sets S_θ and $\tilde{S}_\theta$*

Applying the Theorems 3.5, 3.3 and 4.3, 4.2 one can prove parts (a) and (b) respectively of the following theorem. Here H is as in Section 1.3.

THEOREM 5.2 *Let $0 \leqslant \theta \leqslant 1$. Then*
(a) *$H \subset S_\theta$ if and only if $\theta = 1$,*
(b) *$H \subset \tilde{S}_\theta$ if and only if $\frac{1}{2} \leqslant \theta \leqslant 1$.*

In Figs 1, 2, and 3 we visualize the stability regions S_θ for $\theta = 0,\ \frac{1}{2},\ 1$ by displaying the corresponding sets $R_\theta(y)$. Figure 1 is based on Theorems 3.5 and 4.3. It illustrates the implications

$$|x+1| < 1 - |y| \quad \Rightarrow \quad (x, y) \in S_0 = \tilde{S}_0 \quad \Rightarrow \quad |x+1| \leqslant 1 - |y|\,.$$

Figure 2 is based on Theorems 3.5 and 4.3, and is in accordance with

$$|x-1| > 1 + |y| \quad \Rightarrow \quad (x, y) \in S_1 = \tilde{S}_1 \quad \Rightarrow \quad |x-1| \geqslant 1 + |y|\,.$$

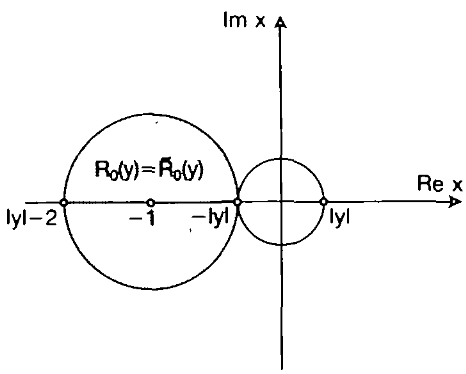

FIG. 1. The stability regions for $\theta = 0$ (cf. (1.8)).

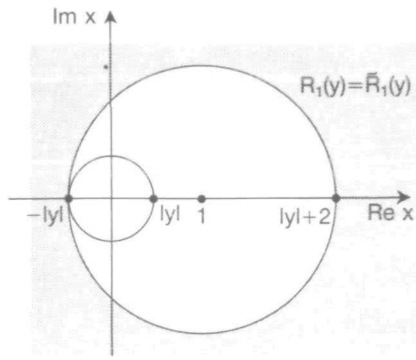

FIG. 2. The stability regions for $\theta = 1$ (cf. (1.8)).

Applying Theorems 3.5 and 3.3 one can prove the implications

$$0 \leq |y| < 2, \quad \mathrm{Re}\, x < -\left\{1 + \frac{(\mathrm{Im}\, x)^2}{4 - |y|^2}\right\}^{\frac{1}{2}} |y| \quad \Rightarrow \quad (x, y) \in S_{\frac{1}{2}},$$

$$(x, y) \in S_{\frac{1}{2}} \quad \Rightarrow \quad 0 \leq |y| \leq 2, \qquad \mathrm{Re}\, x \leq -\left\{1 + \frac{(\mathrm{Im}\, x)^2}{4 - |y|^2}\right\}^{\frac{1}{2}} |y|$$

(with the conventions $a/0 = 0$ (for $a = 0$), $a/0 = \infty$ (for $a \neq 0$)). Figure 3a gives a corresponding illustration.

Figure 3b is based on Theorems 4.3 and 4.2. It illustrates the implications

$$\mathrm{Re}\, x < -|y| \quad \Rightarrow \quad (x, y) \in \tilde{S}_{\frac{1}{2}} \quad \Rightarrow \quad \mathrm{Re}\, x \leq -|y|\,.$$

We thus see that in the important case $\theta = \frac{1}{2}$, when both methods (1.2a) and (1.2b) have an order of accuracy equal to 2, the stability regions S_θ and $\tilde{S}_\theta$ are remarkably different. In particular, $H \not\subset S_{\frac{1}{2}}$ whereas $H \subset \tilde{S}_{\frac{1}{2}}$, as already noted in Section 1.3.

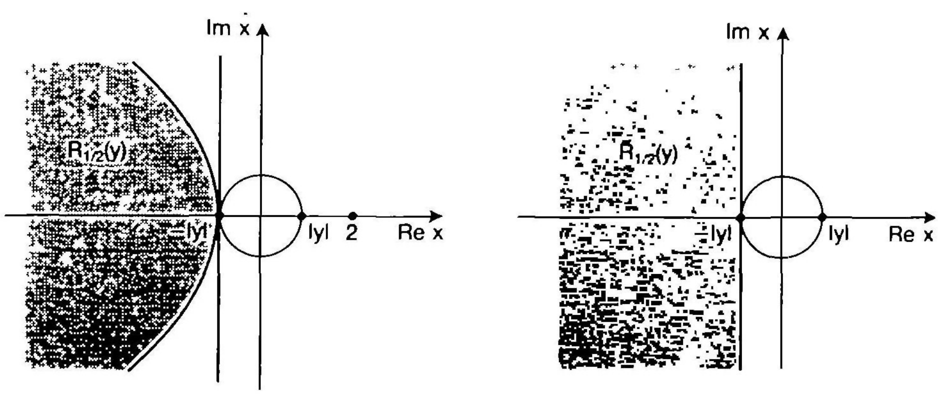

FIG. 3. (a) The stability region S_θ for $\theta = \frac{1}{2}$ (cf. (1.8a)). (b) The stability region $\tilde{S}_\theta$ for $\theta = \frac{1}{2}$ (cf. (1.8b)).

5.3 *A remark about Theorems* 3.3 *and* 4.2

None of the inclusions in Theorems 3.3 and 4.2 can, in general, be replaced by equalities, i.e. we can have

$$A(y,\delta)\subsetneq R_\theta(y,\delta)\subsetneq B(y,\delta),\qquad \tilde A(y,\delta)\subsetneq \tilde R_\theta(y,\delta)\subsetneq \tilde B(y,\delta).$$

We illustrate this fact with $\theta=\delta=0$ and $y=\lambda e^{i\beta}$ where $0<\lambda<1$ and β is real. In view of Corollary 3.4 we have $x\in A(y,0)=\tilde A(y,0)$ if and only if $|x+1|<1-|y|$. Moreover, $x\in B(y,0)=\tilde B(y,0)$ if and only if $|x+1|\leqslant 1-|y|$.

From Theorem 3.3 it follows, by a straightforward calculation, that $x\in R_\theta(y,0)-A(y,0)$ if and only if

$$x=-1+(1-\lambda)e^{i\alpha}\quad\text{with real } \alpha\neq(\beta+2k\pi)(m+1)^{-1}$$

for all $k=0,\pm1,\pm2,\ldots$ and $m=1,2,3,\ldots$.

For instance, if $y=\frac{1}{2}$ the set $R_\theta(y,0)-A(y,0)$ contains all points

$$x=-1+\tfrac{1}{2}\exp\left[\frac{p}{q}i\right]$$

but none of the points

$$x=-1+\tfrac{1}{2}\exp\left[\frac{2p\pi}{q+1}i\right],$$

where p and q are arbitrary positive integers. Consequently, there are an infinite number of points $x\in R_\theta(y,0)-A(y,0)$ and also an infinite number of points $x\in B(y,0)-R_\theta(y,0)$. Since $\tilde R_\theta(y,0)=R_\theta(y,0)$ the conjecture of [4], mentioned in Section 1.3, has thus been disproved.

5.4 *Numerical experiments*

In order to give a numerical illustration to the above theorems, we consider the following two problems, with different stiffness features,

$$U'(t)=-50U(t)+40U(t-1)\quad(t\geqslant0),\qquad U(t)=e^{-t}\quad(t\leqslant0),\tag{5.4}$$

$$U'(t)=-500U(t)+400U(t-1)\quad(t\geqslant0),\qquad U(t)=e^{-t}\quad(t\leqslant0).\tag{5.5}$$

Following [16] we shall compare various numerical processes for approximating the true solutions at $t=10$, which are equal to $U(10)\simeq0.131\,413\,236\,978$ (for (5.4)) and $U(10)\simeq0.109\,545\,478\,582$ (for (5.5)).

We deal with two different meshes, which were also used by Zennaro [15, 16]. The first mesh consists of all gridpoints $t_n=nh\in[0,10]$ with $h=1/M$ and fixed integer $M\geqslant2$. The second mesh contains all integers j with $0\leqslant j\leqslant10$ and, additionally, in each interval $(j-1,j)$ with $1\leqslant j\leqslant10$ it contains all points t satisfying

$$t=[k+j(j+1)/22]h\quad\text{for some integer } k.$$

Here $h=1/M$ as above. The first mesh is called a *constrained mesh* and the second one a *free mesh*.

Both meshes satisfy the natural requirement that the so-called breaking points $0, 1, \ldots, 10$, at which derivatives of the true solutions $U(t)$ may fail to exist, are included. In the constrained mesh the delay $\tau = 1$ is an integer multiple of the stepsize so that only the value $\delta = 0$ (cf. (1.7)) is relevant here. But, in the free mesh, a series of different values for δ are relevant, since in each interval $(j-1, j)$ the uniform mesh with steplength $h = 1/M$ is shifted with respect to the mesh of the preceding interval $(j-2, j-1)$ by a quantity δh with $\delta = j/11$. The free mesh is useful (cf. [15, 16]) in miming a mesh with variable stepsizes as might occur when stepsize control techniques are employed. Moreover, the free mesh will illustrate below the fact that the stability considerations of the above sections are relevant also in situations where the assumptions in the above stability analysis are not completely fulfilled. Applying the θ-methods to the problems (5.4) and (5.5) on the constrained mesh, we arrive at numerical procedures of type (1.6) with $m = M$ and $\delta = 0$. For the methods (1.2a,b) we thus may expect a stable behaviour if

$$(x, y) \in S_{\theta,0}, \qquad (x, y) \in \tilde{S}_{\theta,0}, \tag{5.6a,b}$$

respectively. Here $(x, y) = (-50h, 40h)$ for problem (5.4) and $(x, y) = (-500h, 400h)$ for (5.5).

The obvious variable stepsize versions of the θ-methods on the free mesh amount on each interval $(j-1, j)$ with $2 \leqslant j \leqslant 10$ to $(M-1)$ applications of the procedures (1.6) with $h = 1/M$, $m = M+1$, and $\delta = j/11$. Consequently, we may expect that the conditions

$$(x, y) \in S_\theta = S_{\theta,1-\theta}, \qquad (x, y) \in \tilde{S}_\theta = \tilde{S}_{\theta,0} \tag{5.7a,b}$$

are sufficient for guaranteeing a stable behaviour of the one-leg θ-method and the linear θ-method respectively. Here (x, y) is as above.

As noted above, both the constrained mesh and the free mesh include the breaking points $0, 1, \ldots, 10$. Further, for a given $M \geqslant 2$, the average distance between consecutive gridpoints in the free mesh is *smaller* than in the constrained mesh. Therefore, the free mesh may be expected to yield slightly more-accurate numerical approximations than the constrained mesh, unless the error propagation (stability) on the former mesh is worse than on the latter.

In Table 1 we have listed the absolute values of errors that are present at $t = 10$ in the numerical solution of problem (5.4) when $\theta = \frac{1}{2}$. The two columns in the table pertinent to method (1.2a) refer to the same numerical experiments as reported by Zennaro in [15]. These columns have been included here mainly for comparing them with the corresponding columns pertinent to method (1.2b).

From Theorem 3.3, Corollary 3.4, and Theorems 3.5, 4.2, and 4.3, it follows that (5.6a,b) and (5.7b) are fulfilled, in the situation of Table 1, for all $M \geqslant 2$, whereas stability condition (5.7a) (for the one-leg θ-method (1.2a) on the free mesh) is fulfilled only for $M > 20$. In the table one sees a clear discrepancy between the error in method (1.2a) using a free mesh and the errors in the other three numerical procedures considered. Note that for increasing M this phenomenon becomes less pronounced in particular for $M \geqslant 20$. In view of the above considerations we conclude that this discrepancy for $M < 20$, is due to different

TABLE 1
Numerical results with $\theta = \frac{1}{2}$ for problem (5.4)

M	$h = \frac{1}{M}$	Method (1.2a) $\theta = \frac{1}{2}$ Constrained mesh error	Method (1.2a) $\theta = \frac{1}{2}$ Free mesh error	Method (1.2b) $\theta = \frac{1}{2}$ Constrained mesh error	Method (1.2b) $\theta = \frac{1}{2}$ Free mesh error
2	0·500	1·83E − 01	1·69E + 00	1·89E − 01	1·04E − 02
3	0·333	5·72E − 02	8·53E + 00	5·65E − 02	1·89E − 02
4	0·250	5·45E − 03	3·36E + 00	4·45E − 03	1·98E − 04
5	0·200	8·97E − 03	5·06E − 01	9·62E − 03	2·33E − 03
10	0·100	9·98E − 04	8·47E − 02	1·16E − 03	8·16E − 04
20	0·050	4·66E − 05	2·05E − 04	5·59E − 06	2·51E − 04
50	0·020	7·46E − 06	3·20E − 05	8·94E − 07	4·34E − 05
100	0·010	1·87E − 06	7·90E − 06	2·24E − 07	1·10E − 05
200	0·005	4·67E − 07	1·96E − 06	5·59E − 08	2·78E − 06

stability behaviours of the numerical processes, and that it confirms the above theorems.

Table 2 deals with the same methods as Table 1, but it concerns the numerical solution of (5.5) instead of (5.4). Applying Theorems 3.3, 3.5, 4.2, 4.3, and Corollary 3.4 it can be seen that also here (5.6a,b) and (5.7b) are fulfilled for all $M \geq 2$, whereas (5.7a) is *not* fulfilled for $M \leq 200$. We note the surprising superiority of the linear method (1.2b) over the one-leg method (1.2a) on a free mesh with $M \simeq 20$.

Table 3 concerns the θ-methods with $\theta = \frac{3}{4}$ in the numerical solution of problem (5.5). Similarly, as above, it can be seen that also here (5.6a,b) and (5.7b) are fulfilled for all $M \geq 2$, whereas (5.7a) is fulfilled for $M = 100$, 200 but not for $M \leq 50$. In the table we see again that method (1.2a) on a free mesh

TABLE 2
Numerical results with $\theta = \frac{1}{2}$ for problem (5.5)

M	$h = \frac{1}{M}$	Method (1.2a) $\theta = \frac{1}{2}$ Constrained mesh error	Method (1.2a) $\theta = \frac{1}{2}$ Free mesh error	Method (1.2b) $\theta = \frac{1}{2}$ Constrained mesh error	Method (1.2b) $\theta = \frac{1}{2}$ Free mesh error
2	0·500	2·44E + 00	2·84E − 01	2·51E + 00	6·59E − 01
3	0·333	3·67E − 01	3·19E + 02	3·80E − 01	5·78E − 01
4	0·250	4·40E − 01	2·94E + 01	4·45E − 01	1·88E − 01
5	0·200	1·98E − 01	2·93E + 03	2·00E − 01	1·31E − 01
10	0·100	2·83E − 02	2·41E + 03	2·80E − 02	3·47E − 03
20	0·050	5·97E − 03	2·67E + 03	6·00E − 03	1·38E − 04
50	0·020	6·40E − 06	1·08E + 00	9·20E − 07	1·95E − 05
100	0·010	1·39E − 06	6·80E − 06	1·83E − 08	6·97E − 06
200	0·005	3·47E − 07	1·67E − 06	4·57E − 09	2·12E − 06

TABLE 3
Numerical results with $\theta = \frac{3}{4}$ for problem (5.5)

		Method (1.2a) $\theta = \frac{3}{4}$		Method (1.2b) $\theta = \frac{3}{4}$	
M	$h = \frac{1}{M}$	Constrained mesh error	Free mesh error	Constrained mesh error	Free mesh error
2	0·500	1·42E − 02	4·26E − 02	1·17E − 02	3·53E − 03
3	0·333	4·21E − 04	5·65E − 02	1·63E − 03	2·66E − 04
4	0·250	3·07E − 04	6·65E − 02	3·61E − 04	5·62E − 04
5	0·200	5·36E − 04	3·96E − 02	1·13E − 04	3·05E − 04
10	0·100	5·06E − 05	5·15E − 04	5·39E − 05	1·50E − 04
20	0·050	1·43E − 06	8·28E − 05	2·73E − 05	7·08E − 05
50	0·020	6·84E − 06	5·54E − 06	1·10E − 05	2·55E − 05
100	0·010	4·46E − 06	1·79E − 06	5·48E − 06	1·11E − 05
200	0·005	2·49E − 06	1·96E − 06	2·74E − 06	4·63E − 06

produces less-accurate approximations than the other three numerical procedures, in particular for small M.

For $\delta = 0$, the recursions (1.6a,b) are identical. Still, in case of a constrained mesh, the errors for the methods (1.2a,b) are not identical in the Tables 1, 2, and 3. This can be explained by the fact that $u_1, u_2, \ldots, u_m$ are computed from (1.2a) or (1.2b), but *not* from (1.6a) or (1.6b).

The above numerical experiments are in agreement with the theorems in this paper. Moreover, they support the expectation, based, for example, on Theorem 5.1, that the linear θ-methods (1.2b) may be superior to the one-leg θ-methods (1.2a) in the numerical solution of classes of stiff problems (1.1) when the stepsizes or the delays are not constant.

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