

Superposition in Linear and Nonlinear Ordinary Differential Equations

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1. INTRODUCTION

In [1] Jones and Ames introduced a concept of (nonlinear) superposition for ordinary and partial differential equations.

The purpose of the present paper is to investigate the superposition possibilities for ordinary differential equations of order $p \geq 2$.

The first theorem of this paper (Theorem 1 of Section 3.1) implies that (for the class of ordinary differential equations considered) linear superposition is the only kind of superposition which is possible. However, the differential equations for which this superposition is possible may be nonlinear (cf., Chap. 5).

In Theorems 2, 3, 4, and 5 we shall deal with some consequences of Theorem 1.

In Chap. 5, we shall discuss some examples and applications of the theorems of this paper.

2. PRELIMINARIES

2.1. *The Differential Equation*

In this section we shall specify the class of differential equations to be investigated in this paper.

Let a and b be constants with $-\infty \leq a < b \leq \infty$ and let p and q be integers, $p \geq 2$, $q = p - 1$. The set V is defined by

$$V = \{(t, x_0, x_1, \dots, x_q) : a < t < b, -\infty < x_j < \infty (j = 0, 1, \dots, q)\}.$$

Throughout this paper we assume that f is a function satisfying condition (2.1):

$$f \text{ is a real-valued function, defined and continuous on the set } V. \quad (2.1)$$

We shall deal with the differential equation

$$u^{(p)}(t) = f[t, u(t), u^{(1)}(t), \dots, u^{(q)}(t)], \tag{2.2}$$

$u^{(j)}(t)$ denoting the j th derivative of the function $u(t)$.

The function $M(t, y, \zeta)$ is defined by

$$M(t, y, \zeta) = \max\{|f(t, y, x_1, \dots, x_q)| : |x_j| \leq \zeta^j (j = 1, 2, \dots, q)\}$$

(for $a < t < b$, $-\infty < y < \infty$ and $0 < \zeta < \infty$). In Chap. 3 we shall assume that the function M satisfies

$$\lim_{\zeta \rightarrow \infty} \zeta^{-p} \cdot M(t, y, \zeta) = 0 \tag{2.3}$$

(for all t and y with $a < t < b$, $-\infty < y < \infty$).

Thus (for instance), condition (2.3) is amply satisfied if $f(t, y, x_1, \dots, x_q)$ is a bounded function of (x_1, \dots, x_q) for any fixed t and y .

In Section 3.2 and Chap. 4, we shall assume that f satisfies [in addition to (2.3)]

$$\lim_{\zeta \rightarrow \pm\infty} \zeta^{-1} \cdot f(t, x_0, \dots, x_{q-1}, \zeta) = 0 \tag{2.4}$$

(for all t and x_j with $a < t < b$, $-\infty < x_j < \infty$, $j = 0, 1, \dots, q - 1$).

Many differential equations of type (2.2), encountered in applied mathematics, satisfy (2.3) or even (2.3) and (2.4) (cf. e.g., Section 5.2, for Eq. (2.2), where f does not satisfy (2.3), cf. Section 5.1.A).

2.2. Connecting Functions

In this section we shall introduce a (nonlinear) superposition principle for the differential Eq. (2.2) which is based on the subsequent definition of a connecting function (cf. [1]).

Let n be an integer ≥ 1 and E a subset of

$$\{(t, y_1, y_2, \dots, y_n) : a < t < b, -\infty < y_i < \infty (i = 1, 2, \dots, n)\}$$

satisfying conditions (2.5), (2.6):

$$E \text{ is an open set,} \tag{2.5}$$

$$\left. \begin{aligned} &\text{for any } t_0 \in (a, b) \text{ the set} \\ &\{(y_1, y_2, \dots, y_n) : (t_0, y_1, \dots, y_n) \in E\} \\ &\text{is nonempty and connected.} \end{aligned} \right\} \tag{2.6}$$

DEFINITION 1. F is a *connecting function* of order n for the differential Eq. (2.2) if the following conditions A, B, and C are fulfilled:

(A) F is a real-valued function defined on E with continuous partial derivatives of order p .

(B) None of the derivatives $(\partial/\partial y_i)F(t, y_1, \dots, y_n)$ ($i = 1, 2, \dots, n$) vanishes identically on E .

(C) Whenever $u_1(t), u_2(t), \dots, u_n(t)$ are functions satisfying (2.2) for $t \in$ some interval $(a_1, b_1) \subset (a, b)$ and $[t, u_1(t), \dots, u_n(t)] \in E$ for $a_1 < t < b_1$, then

$$u(t) = F[t, u_1(t), \dots, u_n(t)]$$

is also a solution to (2.2) for $a_1 < t < b_1$.

From this definition it is clear that the existence of a (nonlinear) connecting function implies the possibility of (nonlinear) superposition.

DEFINITION 2. A connecting function F is *linear* if

$$F(t, y_1, \dots, y_n) \equiv \gamma(t) + \sum_{i=1}^n \alpha_i(t) \cdot y_i, \quad (2.7)$$

$\gamma(t), \alpha_1(t), \dots, \alpha_n(t)$ denoting functions defined on (a, b) which do not depend on y_1, \dots, y_n .

DEFINITION 3. F is a *linear* connecting function *with constant coefficients* if F is a connecting function satisfying

$$F(t, y_1, \dots, y_n) \equiv \gamma(t) + \sum_{i=1}^n \alpha_i \cdot y_i, \quad (2.8)$$

$\alpha_1, \alpha_2, \dots, \alpha_n$ denoting real constants and $\gamma(t)$ denoting a fixed function defined on (a, b) .

2.3. Outline of the Following Chapters

In Section 3.1 we shall show that any connecting function for the differential Eq. (2.2), where f satisfies (2.3), is linear. In Section 3.2 we shall show that any connecting function for (2.2), where f satisfies (2.3), as well as (2.4), is a linear connecting function with constant coefficients.

In Chap. 4 (Theorems 3, 4), we shall investigate what conditions on f are necessary and sufficient in order that (2.2) has a linear connecting function with constant coefficients. From Theorem 4 it is then deduced (cf. Theorem 5) which differential Eq. (2.2), where f satisfies (2.3) and (2.4), have a connecting function of order $n > 1$.

In Section 5.1 we present some counterexamples which show that the conditions (2.3), (2.4), respectively, imposed on f in the theorems of Chap. 3,

cannot be omitted. In 5.2 we deal with the connecting functions for Duffing's equation. In 5.3 we mention some problems appearing in other areas of mathematics which are related to the topics of this paper.

3. THE FORM OF THE CONNECTING FUNCTIONS

3.1. The Differential Equation (2.2), Where f Satisfies (2.3)

We shall prove the following theorem:

THEOREM 1. *Let f satisfy (2.1), (2.3) and let F be a connecting function for the differential Eq. (2.2). Then F is a linear connecting function.*

Proof. Let $F(t, y_1, \dots, y_n)$ be a connecting function for (2.2). We shall show that F is linear.

(A) Let (t_0, y_1, \dots, y_n) be a fixed point of E . Let $x_{i,j}$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, q$) be arbitrary real numbers. In view of (2.1), (2.5) (cf., e.g., [2]) there are numbers a_1, b_1 with $a \leq a_1 < t_0 < b_1 \leq b$ and functions $u_1(t), \dots, u_n(t)$ satisfying

$$\left. \begin{aligned} u_i(t_0) &= y_i, \\ u_i^{(j)}(t_0) &= x_{i,j} \quad (j = 1, 2, \dots, q), \\ u_i^{(p)}(t) &= f(t, u_i(t), u_i^{(1)}(t), \dots, u_i^{(q)}(t)) \quad (a_1 < t < b_1), \quad (i = 1, 2, \dots, n), \end{aligned} \right\} \quad (3.1)$$

where the points $[t, u_1(t), \dots, u_n(t)]$ remain in E for $a_1 < t < b_1$.

The function $u(t)$ is defined by

$$u(t) = F(t, u_1(t), \dots, u_n(t)), \quad a_1 < t < b_1. \quad (3.2)$$

Since F is a connecting function (cf. Definition 1), the function $u(t)$ satisfies:

$$u^{(p)}(t) = f(t, u(t), u^{(1)}(t), \dots, u^{(q)}(t)), \quad a_1 < t < b_1. \quad (3.3)$$

Using (3.2), we shall express [in (B)] the derivatives $u^{(r)}(t)$ ($1 \leq r \leq p$) in the partial derivatives of F and the derivatives of $u_1(t), u_2(t), \dots, u_n(t)$. In (C) we shall substitute the expressions obtained for $u^{(r)}(t)$ into formula (3.3). Using the resultant formula and (2.3) we then prove (in (D)) that F is linear.

(B) We define $u_0(t) = t$ (for $a_1 < t < b_1$). Hence [cf. (3.2)], $u(t) = F[u_0(t), u_1(t), \dots, u_n(t)]$. Applying the chain rule for a function of several variables (cf. [3]), we get for $1 \leq r \leq p$

$$u^{(r)}(t) = \sum_A \left\{ \beta(A) \cdot \left(\prod_{i=0}^n D_i^{b(i)} \right) F \cdot \prod_{i=0}^n \prod_{j=1}^r (D^j u_i)^{a(i,j)} \right\} \quad (3.4)$$

(for $a_1 < t < b_1$), where the summation extends over all $(n + 1)$ by r matrices $A = [a(i, j)]$; the entries $a(i, j)$ ($i = 0, 1, \dots, n$; $j = 1, 2, \dots, r$) of which are integers ≥ 0 satisfying

$$\sum_{i=0}^n \sum_{j=1}^r j \cdot a(i, j) = r. \quad (3.4a)$$

The exponents $b(i)$ are defined by

$$b(i) = \sum_{j=1}^r a(i, j), \quad (3.5)$$

and the operators D_i, D by

$$D_i F = \frac{\partial}{\partial u_i} F, \quad D u_i = \frac{d}{dt} u_i \quad (i = 0, 1, \dots, n).$$

It may be shown that the coefficients $\beta(A)$ appearing in (3.4) satisfy

$$\beta(A) > 0$$

(cf. [3]).

Taking $t = t_0$ in (3.4), we get (for $1 \leq r \leq p$)

$$u^{(r)}(t_0) = \sum_A \delta(A) \cdot \prod_{i=0}^n \prod_{j=1}^r (D^j u_i(t_0))^{a(i, j)}, \quad (3.6)$$

where

$$\delta(A) = \beta(A) \cdot \left(\prod_{i=0}^n D_i^{b(i)} \right) F(t_0, y_1, \dots, y_n). \quad (3.7)$$

Since $D^1 u_0(t_0) = 1$, $D^j u_0(t_0) = 0$ ($2 \leq j \leq p$), and $D^j u_i(t_0) = x_{i, j}$ ($1 \leq i \leq n$, $1 \leq j \leq q$), formula (3.6) for $1 \leq r \leq q$ and for $r = p$ is equivalent to (3.8), (3.10), respectively:

$$u^{(r)}(t_0) = P_r \quad (1 \leq r \leq q), \quad (3.8)$$

where

$$P_r = P_r(x_{i, j}) = \sum_A \delta(A) \cdot \prod_{i=1}^n \prod_{j=1}^r (x_{i, j})^{a(i, j)}, \quad (3.9)$$

the summation in (3.9) being for all matrices $A = [a(i, j)]$, the entries $a(i, j)$ of which are integers ≥ 0 satisfying (3.4a) with

$$\begin{aligned} a(0, 2) = a(0, 3) = \dots = a(0, r) &= 0. \\ u^{(p)}(t_0) &= P + Q + R, \end{aligned} \quad (3.10)$$

where P, Q and R are defined by (3.11), (3.12), (3.13), respectively:

$$P = P(x_{i,j}) = \sum_A \delta(A) \cdot \prod_{i=1}^n \prod_{j=1}^q (x_{i,j})^{a(i,j)}, \tag{3.11}$$

where $A = [a(i, j)]$ and the integers $a(i, j) \geq 0$ ($i = 0, 1, \dots, n; j = 1, 2, \dots, p$) are subject to the constraints:

$$\left. \begin{aligned} a(0, j) = a(i, p) = 0 \quad (1 \leq j \leq p, 0 \leq i \leq n), \\ \sum_{i=1}^n \sum_{j=1}^q j \cdot a(i, j) = p. \end{aligned} \right\} \tag{3.11a}$$

$$Q = Q(x_{i,j}) = \sum_A \delta(A) \prod_{i=1}^n \prod_{j=1}^q (x_{i,j})^{a(i,j)}, \tag{3.12}$$

where $A = [a(i, j)]$ and the integers $a(i, j) \geq 0$ ($i = 0, 1, \dots, n; j = 1, 2, \dots, p$) are subject to

$$\left. \begin{aligned} a(0, 1) > 0, \quad a(0, j) = a(i, p) = 0 \quad (2 \leq j \leq p, 0 \leq i \leq n), \\ \sum_{i=0}^n \sum_{j=1}^q j \cdot a(i, j) = p. \end{aligned} \right\} \tag{3.12a}$$

$$R = \sum_A \delta(A) \cdot D^p u_k(t_0), \tag{3.13}$$

where $A = [a(i, j)]$ and the integers $a(i, j)$ vanish with the exception of one of the integers $a(i, p)$ ($i = 1, 2, \dots, n$) which equals 1 and is denoted by $a(k, p)$.

From (3.13), (3.1) it follows that

$$R = R(x_{i,j}) = \sum_A \delta(A) \cdot f(t_0, y_k, x_{k,1}, \dots, x_{k,q}). \tag{3.14}$$

(C) Substituting the expressions (3.8), (3.10) for the derivatives $u^{(r)}(t_0)$ in formula (3.3), we obtain the equality

$$P(x_{i,j}) + Q(x_{i,j}) + R(x_{i,j}) = f[t_0, u(t_0), P_1(x_{i,j}), \dots, P_q(x_{i,j})].$$

Consequently,

$$|P(x_{i,j})| \leq |Q(x_{i,j})| + |R(x_{i,j})| + |f[t_0, u(t_0), P_1(x_{i,j}), \dots, P_q(x_{i,j})]|. \tag{3.15}$$

Using (3.15), (2.3) we shall show that $P = 0$.

Let $\xi_{i,j}$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, q$) be fixed real numbers and let ζ be a variable > 1 . Applying (3.15) with $x_{i,j} = \zeta^j \cdot \xi_{i,j}$, we get

$$\begin{aligned} |P(\zeta^j \xi_{i,j})| &\leq |Q(\zeta^j \xi_{i,j})| + |R(\zeta^j \xi_{i,j})| \\ &+ |f[t_0, u(t_0), P_1(\zeta^j \xi_{i,j}), \dots, P_q(\zeta^j \xi_{i,j})]|. \end{aligned} \quad (3.16)$$

In view of (3.11), (3.11a), we have

$$P(\zeta^j \xi_{i,j}) = \zeta^p \cdot P(\xi_{i,j}). \quad (3.17)$$

From (3.12), (3.12a), and (3.9) it follows that there is a number $\omega > 1$ (not depending on ζ) such that

$$|Q(\zeta^j \xi_{i,j})| \leq \omega \cdot \zeta^q, \quad (3.18)$$

$$|P_r(\zeta^j \xi_{i,j})| \leq \omega \cdot \zeta^r \quad (1 \leq r \leq q), \quad (3.19)$$

$$|\zeta^r \xi_{k,r}| \leq \omega \cdot \zeta^r \quad (1 \leq r \leq q, 1 \leq k \leq n). \quad (3.20)$$

In view of (3.16), (3.17), (3.18), (3.14), (3.19), we obtain the inequality

$$\begin{aligned} \zeta^p \cdot |P(\xi_{i,j})| &\leq \omega \cdot \zeta^q + O \left\{ \sum_{k=1}^n |f(t_0, y_k, \zeta \xi_{k,1}, \dots, \zeta^q \xi_{k,q})| \right\} \\ &+ |f(t_0, u(t_0), x_1, \dots, x_q)|, \end{aligned} \quad (3.21)$$

where the numbers x_r satisfy $|x_r| \leq \omega \cdot \zeta^r$ ($1 \leq r \leq q$).

Defining

$$N(y, \zeta) = \max\{|f(t_0, y, x_1, \dots, x_q)| : |x_r| \leq \omega \zeta^r \ (1 \leq r \leq q)\}, \quad (3.22)$$

we get from (3.21), (3.20) the inequality

$$\zeta^p \cdot |P(\xi_{i,j})| \leq \omega \cdot \zeta^q + O \left\{ \sum_{k=1}^n N(y_k, \zeta) \right\} + N[u(t_0), \zeta]. \quad (3.23)$$

Since $\omega > 1$, we have [cf. (3.22)]

$$\begin{aligned} N(y, \zeta) &\leq \max\{|f(t_0, y, x_1, \dots, x_q)| : |x_r| \leq \omega^r \zeta^r \ (1 \leq r \leq q)\} \\ &= M(t_0, y, \omega \zeta), \end{aligned}$$

where M is the function appearing in (2.3). In view of (2.3) it follows that $\zeta^{-p} \cdot N(y, \zeta) \rightarrow 0$ for $\zeta \rightarrow \infty$. Dividing both sides of (3.23) by ζ^p , we thus get

$$|P(\xi_{i,j})| \leq \omega \cdot \zeta^{-1} + o(1) + o(1) = o(1) \quad (\text{for } \zeta \rightarrow \infty).$$

Hence, $P(\xi_{i,j}) = 0$. It follows that the polynomial $P(x_{i,j})$ vanishes identically.

(D) Since $P = 0$, the coefficients $\delta(A)$ in formula (3.11) vanish. In view of $\beta(A) \neq 0$, we may conclude [cf. (3.7)] that

$$\left(\prod_{i=0}^n D_i^{b(i)} \right) F(t_0, y_1, \dots, y_n) = 0, \quad (3.24)$$

provided the exponents $b(i)$ are defined by (3.5) and the $a(i, j)$ ($i = 0, 1, \dots, n; j = 1, 2, \dots, p$) are integers ≥ 0 satisfying (3.11a).

As is easily verified any second order derivative $D_k D_m F(t_0, y_1, \dots, y_n)$ ($k, m = 1, 2, \dots, n$) is of the form appearing at the left side of the equality (3.24). Consequently,

$$D_k D_m F(t_0, y_1, \dots, y_n) = 0 \quad (k, m = 1, 2, \dots, n). \quad (3.25)$$

From (2.5), (2.6) it follows that $E_0 = \{(y_1, y_2, \dots, y_n) : (t_0, y_1, \dots, y_n) \in E\}$ is an open connected subset of the n -dimensional Cartesian space R_n . The function $F_0(y_1, y_2, \dots, y_n) \equiv F(t_0, y_1, \dots, y_n)$ is defined and two times continuously differentiable on E_0 with vanishing derivatives of order 2 [cf. (3.25)]. It follows (cf. e.g. [4]) that $F_0(y_1, y_2, \dots, y_n)$ is a linear function of the variables y_i . Consequently, $F(t, y_1, \dots, y_n)$ satisfies (2.7) and the theorem is proved.

3.2. The Differential Equation (2.2), Where f Satisfies (2.3), (2.4)

In this section we shall give a stronger version of Theorem 1 assuming that condition (2.4) is satisfied.

THEOREM 2. *Let f satisfy (2.1), (2.3), (2.4), and let F be a connecting function for the differential Eq. (2.2). Then F is a linear connecting function with constant coefficients.*

Proof. Let F denote a connecting function for (2.2). By virtue of Theorem 1, F is linear, i.e.,

$$F(t, y_1, \dots, y_n) \equiv \gamma(t) + \sum_{i=1}^n \alpha_i(t) \cdot y_i. \quad (3.26)$$

Let k be an integer with $1 \leq k \leq n$. We shall show that the function $\alpha_k(t)$ is constant on (a, b) .

(A) Since (3.26) holds for all $(t, y_1, \dots, y_n) \in$ the open set E , and F has a continuous derivative of order p with respect to t (cf. condition A in Section 2.2), the functions $\alpha_i(t)$, $\gamma(t)$ are also p times continuously differentiable on (a, b) .

Let (t_0, y_1, \dots, y_n) be a fixed point of E and let $x_{i,j} = 0$ [$i = 1, 2, \dots, n; j = 1, 2, \dots, q; (i, j) \neq (k, q)$], $x_{k,q} = \zeta$, ζ denoting a real variable > 0 .

We now proceed exactly as in part A of the proof of Theorem 1 [cf. (3.1), (3.2), (3.3)]. In order to use (3.3), we need an appropriate expression for $u^{(r)}(t)$ ($a_1 < t < b_1$, $1 \leq r \leq p$). Using (3.2), (3.26) and Leibniz' theorem, we obtain (for $1 \leq r \leq p$)

$$\begin{aligned} u^{(r)}(t) &= D^r F(t, u_1(t), \dots, u_n(t)) \\ &= D^r \left[\gamma(t) + \sum_{i=1}^n \alpha_i(t) \cdot u_i(t) \right] \\ &= \sum_{i=1}^n \sum_{j=0}^r \binom{r}{j} \cdot [D^j u_i(t)] \cdot [D^{r-j} \alpha_i(t)] + D^r \gamma(t). \end{aligned}$$

In view of (3.1) and the above definition of $x_{i,j}$, we thus get

$$u^{(r)}(t_0) = \sum_{i=1}^n y_i \cdot D^r \alpha_i(t_0) + D^r \gamma(t_0) \quad (1 \leq r < q), \quad (3.27)$$

$$u^{(q)}(t_0) = \sum_{i=1}^n y_i \cdot D^q \alpha_i(t_0) + \zeta \cdot \alpha_k(t_0) + D^q \gamma(t_0), \quad (3.28)$$

$$\begin{aligned} u^{(p)}(t_0) &= \sum_{i=1}^n y_i \cdot D^p \alpha_i(t_0) + p \cdot \zeta \cdot D \alpha_k(t_0) \\ &\quad + \sum_{i=1}^n f(t_0, y_i, 0, \dots, 0, \delta_{i,k} \cdot \zeta) \cdot \alpha_i(t_0) \\ &\quad + D^p \gamma(t_0), \end{aligned} \quad (3.29)$$

$\delta_{i,k}$ denoting the Kronecker delta.

(B) From (3.3) it follows that

$$\zeta^{-1} \cdot u^{(p)}(t_0) = \zeta^{-1} \cdot f[t_0, u(t_0), u^{(1)}(t_0), \dots, u^{(q)}(t_0)]. \quad (3.30)$$

Substituting the expression (3.29) for $u^{(p)}(t_0)$ in (3.30), we get

$$\begin{aligned} p \cdot D \alpha_k(t_0) + \zeta^{-1} \cdot f(t_0, y_k, 0, \dots, 0, \zeta) \cdot \alpha_k(t_0) + O(\zeta^{-1}) \\ = \zeta^{-1} \cdot f[t_0, u(t_0), u^{(1)}(t_0), \dots, u^{(q)}(t_0)]. \end{aligned}$$

In view of (2.4) this yields

$$p \cdot D \alpha_k(t_0) = \lim_{\zeta \rightarrow \infty} \zeta^{-1} f[t_0, u(t_0), u^{(1)}(t_0), \dots, u^{(q)}(t_0)]. \quad (3.31)$$

(1) Assume $\alpha_k(t_0) = 0$. Then $u^{(j)}(t_0)$ is independent of ζ [for $1 \leq j \leq q$; cf. (3.27), (3.28)]. Hence, the limit at the right side of (3.31) equals zero. It follows that $D\alpha_k(t_0) = 0$.

(2) Assume $\alpha_k(t_0) \neq 0$. Then

$$\begin{aligned} & \lim_{\zeta \rightarrow \infty} \zeta^{-1} f[t_0, u(t_0), u^{(1)}(t_0), \dots, u^{(q)}(t_0)] \\ &= \lim_{\zeta \rightarrow \infty} [\zeta^{-1} \cdot u^{(q)}(t_0)] \cdot [u^{(q)}(t_0)]^{-1} \cdot f[t_0, u(t_0), u^{(1)}(t_0), \dots, u^{(q)}(t_0)] \\ &= \alpha_k(t_0) \cdot \lim_{\zeta \rightarrow \infty} [u^{(q)}(t_0)]^{-1} \cdot f[t_0, u(t_0), \dots, u^{(q)}(t_0)] = 0 \end{aligned}$$

[cf. (3.27), (3.28), (2.4)]. Consequently, [cf. (3.31)] $D\alpha_k(t_0) = 0$.

Since $D\alpha_k(t_0) = 0$ and t_0 denotes an arbitrary point $\in (a, b)$, it follows that $\alpha_k(t)$ is constant on (a, b) . Hence, F is of the form (2.8).

4. CONDITIONS ON f WHICH ARE NECESSARY AND SUFFICIENT FOR THE EXISTENCE OF A CONNECTING FUNCTION

4.1. *Connecting Functions of Order $n \geq 1$*

In this section we study the differential equations of type (2.2) [where f satisfies (2.1)] for which a linear connecting function F with constant coefficients exists.

Let $n \geq 1$ and let the function F be defined by

$$F(t, y_1, \dots, y_n) = \gamma(t) + \sum_{i=1}^n \alpha_i \cdot y_i \tag{4.1}$$

[for $(t, y_1, \dots, y_n) \in$ the set E of Section 2.2], where none of the constants α_i vanishes and the function $\gamma(t)$ is defined and p times continuously differentiable on (a, b) .

We have the following theorem:

THEOREM 3. *Let f satisfy (2.1). Then the function F defined by (4.1) is a connecting function for the differential equation (2.2) if, and only if, f satisfies the following functional equation*

$$\begin{aligned} & f\left(t, \gamma(t) + \sum_{i=1}^n \alpha_i \cdot y_i, \gamma^{(1)}(t) + \sum_{i=1}^n \alpha_i \cdot x_{i,1}, \dots, \gamma^{(q)}(t) + \sum_{i=1}^n \alpha_i \cdot x_{i,q}\right) \\ &= \gamma^{(p)}(t) + \sum_{i=1}^n \alpha_i \cdot f(t, y_i, x_{i,1}, \dots, x_{i,q}) \end{aligned} \tag{4.2}$$

[for $(t, y_1, \dots, y_n) \in E, -\infty < x_{i,j} < \infty$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, q$)].

Proof. (A) Let F [defined by (4.1)] be a connecting function. We shall show that (4.2) is satisfied.

Let (t_0, y_1, \dots, y_n) be a fixed point of E and let $x_{i,j}$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, q$) be arbitrary real numbers. We now proceed exactly as in part A of the proof of Theorem 1 [cf. (3.1), (3.2), (3.3)]. We have [cf. (3.2), (4.1)]

$$u(t) = \gamma(t) + \sum_{i=1}^n \alpha_i \cdot u_i(t) \quad (a_1 < t < b_1) \quad (4.3)$$

and, consequently,

$$u^{(r)}(t) = \gamma^{(r)}(t) + \sum_{i=1}^n \alpha_i \cdot u_i^{(r)}(t) \quad (0 \leq r \leq p, a_1 < t < b_1). \quad (4.4)$$

Substituting the expression (4.4) for $u^{(r)}(t)$ into (3.3), we get

$$\begin{aligned} & \gamma^{(p)}(t) + \sum_{i=1}^n \alpha_i \cdot u_i^{(p)}(t) \\ &= f \left[t, \gamma^{(0)}(t) + \sum_{i=1}^n \alpha_i \cdot u_i^{(0)}(t), \dots, \gamma^{(a)}(t) + \sum_{i=1}^n \alpha_i \cdot u_i^{(a)}(t) \right]. \end{aligned}$$

Using (3.1) we obtain for $t = t_0$:

$$\begin{aligned} & \gamma^{(p)}(t_0) + \sum_{i=1}^n \alpha_i \cdot f(t_0, y_i, x_{i,1}, \dots, x_{i,q}) \\ &= f \left[t_0, \gamma(t_0) + \sum_{i=1}^n \alpha_i \cdot y_i, \dots, \gamma^{(a)}(t_0) + \sum_{i=1}^n \alpha_i \cdot x_{i,a} \right]. \end{aligned}$$

It follows that (4.2) is satisfied.

(B) Let (4.2) hold. Assume that $u_1(t), u_2(t), \dots, u_n(t)$ are functions satisfying (2.2) for $a_1 < t < b_1$, where $a \leq a_1 < b_1 \leq b$ and $[t, u_1(t), \dots, u_n(t)] \in E$ (for $a_1 < t < b_1$). We define $u(t)$ by (4.3). Using (4.4), (4.2) it follows readily that $u(t)$ satisfies (2.2) for $a_1 < t < b_1$, which completes the proof of the theorem.

4.2. Connecting Functions of Order $n \geq 2$; $E = (a, b) \times R_n$

In this section we assume $n \geq 2$ and

$$\begin{aligned} E &= (a, b) \times R_n \\ &= \{(t, y_1, \dots, y_n) : a < t < b, -\infty < y_i < \infty (i = 1, 2, \dots, n)\}. \end{aligned}$$

We shall fix which equations of type (2.2) have a connecting function of the form (4.1).

THEOREM 4. *Let f satisfy (2.1) and let $n \geq 2$, $E = (a, b) \times R_n$. Then the function F [defined by (4.1)] is a connecting function for the differential equation (2.2) if, and only if, f is linear with respect to x_0, x_1, \dots, x_q , i.e.,*

$$f(t, x_0, \dots, x_q) \equiv \lambda(t) + \sum_{j=0}^q \lambda_j(t) \cdot x_j \tag{4.5}$$

for some functions $\lambda(t), \lambda_j(t)$ defined on (a, b) , and

$$\gamma^{(p)}(t) = \left(1 - \sum_{i=1}^n \alpha_i\right) \cdot \lambda(t) + \sum_{j=0}^q \lambda_j(t) \cdot \gamma^{(j)}(t) \tag{4.6}$$

(for $a < t < b$).

Proof. (A) Let f satisfy (2.1), (4.5) and let F [cf. (4.1)] satisfy (4.6). In order to prove that F is a connecting function for the differential equation (2.2), we only have to show that condition (4.2) is satisfied (cf. Theorem 3).

With $y_i = x_{i,0}$ ($i = 1, 2, \dots, n$) we have [cf. (4.5), (4.6)]

$$\begin{aligned} f\left(t, \gamma(t) + \sum_{i=1}^n \alpha_i y_i, \gamma^{(1)}(t) + \sum_{i=1}^n \alpha_i x_{i,1}, \dots, \gamma^{(q)}(t) + \sum_{i=1}^n \alpha_i x_{i,q}\right) \\ \equiv \lambda(t) + \sum_{j=0}^q \lambda_j(t) \cdot \left[\gamma^{(j)}(t) + \sum_{i=1}^n \alpha_i x_{i,j}\right] \\ \equiv \gamma^{(p)}(t) + \sum_{i=1}^n \alpha_i \cdot \left[\sum_{j=0}^q \lambda_j(t) x_{i,j} + \lambda(t)\right] \\ \equiv \gamma^{(p)}(t) + \sum_{i=1}^n \alpha_i \cdot f(t, y_i, x_{i,1}, \dots, x_{i,q}). \end{aligned}$$

Hence, condition (4.2) is fulfilled and F is a connecting function.

(B) Let F [cf. (4.1)] be a connecting function for the Eq. (2.2). By virtue of Theorem 3 f satisfies the functional Eq. (4.2). Using (4.2) we shall show (in B.1.) that f is linear with respect to x_0, x_1, \dots, x_q . In B.2. we shall prove that (4.6) holds.

B.1. Let ξ_j, η_j denote variables $\in (-\infty, \infty)$ (for $j = 0, 1, \dots, q$) and let t be a fixed number in (a, b) . We define

$$\alpha = \alpha_1, \quad \beta = \alpha_2, \quad \gamma_j = \gamma^{(j)}(t) \quad (0 \leq j \leq q),$$

$$\mu = \gamma^{(p)}(t) + \sum_{i=3}^n \alpha_i \cdot f(t, 0, \dots, 0).$$

Applying (4.2) with

$$y_1 = \xi_0, \quad y_2 = \eta_0, \quad y_i = 0 \quad (2 < i \leq n)$$

$$x_{1,j} = \xi_j, \quad x_{2,j} = \eta_j, \quad x_{i,j} = 0 \quad (2 < i \leq n, 1 \leq j \leq q),$$

we obtain

$$f(t, \alpha\xi_0 + \beta\eta_0 + \gamma_0, \dots, \alpha\xi_q + \beta\eta_q + \gamma_q)$$

$$= \alpha \cdot f(t, \xi_0, \dots, \xi_q) + \beta \cdot f(t, \eta_0, \dots, \eta_q) + \mu. \quad (4.7)$$

Using the vector notation

$$x = (\xi_0, \xi_1, \dots, \xi_q), \quad y = (\eta_0, \eta_1, \dots, \eta_q), \quad c = (\gamma_0, \gamma_1, \dots, \gamma_q)$$

and defining

$$g(x) = g(\xi_0, \xi_1, \dots, \xi_q) = f(t, \xi_0, \dots, \xi_q) \quad (4.8)$$

formula (4.7) may be written as

$$g(\alpha x + \beta y + c) = \alpha g(x) + \beta g(y) + \mu. \quad (4.9)$$

Let $\delta = \beta \cdot \alpha^{-1}$ {in view of $\alpha_1 \neq 0$ [cf. (4.1)], we have $\alpha \neq 0$ }. Since

$$g(\alpha x + \beta y + c) = g(\alpha \cdot (x + \delta y) + \beta \cdot 0 + c) = \alpha g(x + \delta y) + \beta g(0) + \mu$$

[cf. (4.9)] we get in view of (4.9)

$$\alpha g(x + \delta y) + \beta g(0) + \mu = \alpha g(x) + \beta g(y) + \mu,$$

and consequently

$$g(x + \delta y) = g(x) + \delta g(y) - \delta g(0). \quad (4.10)$$

We define the function G by

$$G(x) = g(x) - g(0). \quad (4.11)$$

Hence, (4.10) yields

$$G(x + \delta y) = G(x) + \delta G(y). \quad (4.12)$$

Applying (4.12) with $x = u, y = \delta^{-1} \cdot v$ and $x = 0, y = \delta^{-1} \cdot v$, respectively, we obtain the formulas

$$\begin{aligned} G(u + v) &= G(u) + \delta G(\delta^{-1} \cdot v), \\ G(v) &= G(0) + \delta G(\delta^{-1} \cdot v). \end{aligned}$$

Consequently,

$$G(u + v) = G(u) + G(v) - G(0).$$

In view of $G(0) = 0$ [cf. (4.11)] we have

$$G(u + v) = G(u) + G(v),$$

which is Cauchy-Abel's functional equation (cf. [5]). By virtue of the continuity of G [cf. (4.11), (4.8), (2.1)] it follows (cf. [5]) that $G(x)$ is a linear homogeneous function of the variables $\xi_0, \xi_1, \dots, \xi_q$. Using (4.11), (4.8) we immediately obtain (4.5).

B.2. We shall show that (4.6) is fulfilled. Formula (4.2) with $y_i = x_{i,j} = 0$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, q$) yields in view of (4.5)

$$\lambda(t) + \sum_{j=0}^q \lambda_j(t) \cdot \gamma^{(j)}(t) = \gamma^{(p)}(t) + \sum_{i=1}^n \alpha_i \cdot \lambda(t), \quad a < t < b.$$

Hence, (4.6) is satisfied. This completes the proof of Theorem 4.

Using Theorems 2, 4, it may be fixed which differential equations (2.2), where f satisfies (2.1), (2.3), (2.4), have a connecting function:

THEOREM 5. *Let f satisfy (2.1), (2.3), (2.4), and let $n \geq 2, E = (a, b) \times R_n$. Then the differential Eq. (2.2) has a connecting function if, and only if, $f(t, x_0, \dots, x_q)$ is linear with respect to the variables x_0, x_1, \dots, x_q [i.e. if, and only if, f satisfies (4.5)].*

Proof. (A) Let F be a connecting function for (2.2). In view of Theorem 2, F is a function of type (2.8). It follows from the conditions A, B (cf. Section 2.2, Definition 1), respectively, that $\gamma(t)$ [appearing in (2.8)] has a continuous derivative of order p and that the coefficients α_i [$i = 1, 2, \dots, n$; cf. (2.8)] don't vanish. Hence, the conditions imposed on $\gamma(t), \alpha_i$ in (4.1) are satisfied and we may apply Theorem 4. It follows that (4.5) is satisfied.

(B) Let f satisfy (4.5). Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be arbitrary non vanishing numbers with

$$\sum_{i=1}^n \alpha_i = 1. \tag{4.13}$$

We define

$$F(t, y_1, \dots, y_n) \equiv \sum_{i=1}^n \alpha_i \cdot y_i. \quad (4.14)$$

Hence, F is of the form (4.1) [with $\gamma(t) \equiv 0$] and condition (4.6) is fulfilled. In view of Theorem 4, we may conclude that F is a connecting function for (2.2). This completes the proof of the theorem.

5. EXAMPLES AND CONCLUDING REMARKS

5.1. Counterexamples

(A) The following example shows that condition (2.3) cannot be omitted in Theorem 1 (Section 3.1).

$$p = 2, \quad a = -\infty, \quad b = \infty, \quad f(t, x_0, x_1) = -(x_1)^2, \quad \text{and} \quad n = 2, \\ E = (a, b) \times R_2, \quad F(t, y_1, y_2) = \log[\exp(y_1) + \exp(y_2)].$$

f fails to satisfy (2.3) and F is a nonlinear connecting function for the differential Eq. (2.2).

Using the method of [1], similar counterexamples may be constructed for $p > 2$.

(B) The following example shows that condition (2.4) in Theorem 2 (Section 3.2) is indispensable.

$$p = 2, \quad a = 0, \quad b = \infty, \quad f(t, x_0, x_1) = 0 \quad (\text{for } 0 < t, x_0 < 1), \\ f(t, x_0, x_1) = t^{-1} \cdot (2x_1 - t^{-1} \cdot x_0 \log x_0) \cdot \log x_0 \quad (\text{for } 0 < t, 1 \leq x_0)$$

and

$$n = 1, \quad E = \{(t, y) : 0 < t, 1 < y\}, \quad F(t, y) = \exp(t) \cdot y.$$

f fails to satisfy (2.4) and F may be shown to be a linear connecting function for Eq. (2.2) with a nonconstant coefficient.

(C) The following two examples show that neither the condition $n \geq 2$, nor the condition $E = (a, b) \times R_n$ may be omitted in the theorems of Section 4.2.

$$(1) \quad p \geq 2, \quad a = -\infty, \quad b = \infty, \quad f(t, x_0, \dots, x_q) = 0 \quad (\text{for } x_0 = 0), \\ f(t, x_0, \dots, x_q) = x_0 \cdot \sin(2\pi \cdot {}^2\log |x_0|) \quad (\text{for } x_0 \neq 0)$$

and

$$n = 1, \quad E = (a, b) \times R_1, \quad F(t, y) = 2 \cdot y.$$

$$(2) \quad p \geq 2, \quad a = -\infty, \quad b = \infty, \quad f(t, x_0, \dots, x_q) = |x_0|$$

and

$$n \geq 2, \quad E = \{(t, y_1, \dots, y_n) : a < t < b, 0 < y_i (i = 1, 2, \dots, n)\},$$

$$F(t, y_1, \dots, y_n) = y_1 + y_2 + \dots + y_n.$$

5.2. *Duffing's Equation*

An illustration of the above we investigate whether there exist connecting functions for Duffing's equation without damping term (cf. [6]):

$$u''(t) + u(t) + Q \cdot [u(t)]^3 = R \cdot \cos \omega t \tag{5.1}$$

(ω, Q and R denoting real constants), which is an equation of type (2.2) with $p = 2$ and f satisfying (2.1), (2.3), (2.4). Let $-\infty \leq a < b \leq \infty, n \geq 1, E = (a, b) \times R_n$.

(A) Assume $n \geq 2$. From Theorem 5 it follows that Eq. (5.1) has a connecting function if, and only if, $Q = 0$.

Let $Q = 0$. Then (cf. Theorems 2, 4) any connecting function is of the form (4.1), where $\gamma(t)$ satisfies (4.6) with

$$\lambda(t) = R \cos \omega t, \quad \lambda_0(t) = -1, \quad \lambda_1(t) = 0.$$

(B) Assume $n = 1$. In view of Theorem 2, any (possible) connecting function F is of the form

$$F(t, y) \equiv \alpha \cdot y + \gamma(t), \tag{5.2}$$

α denoting a constant $\neq 0$. From Theorem 3 it follows that a function F satisfying (5.2) is a connecting function for (5.1) if, and only if,

$$- [\alpha y + \gamma(t)] - Q \cdot [\alpha y + \gamma(t)]^3 + R \cos \omega t$$

$$= \gamma''(t) + \alpha \cdot [-y - Qy^3 + R \cos \omega t] \tag{5.3}$$

for $a < t < b, -\infty < y < \infty$. A little calculation shows that (5.3) is equivalent to

$$Q\alpha(\alpha^2 - 1) \cdot y^3 + 3Q\alpha^2\gamma(t) \cdot y^2 + 3Q\alpha[\gamma(t)]^2 \cdot y$$

$$+ \{\gamma''(t) + \gamma(t) + Q[\gamma(t)]^3 + (\alpha - 1) R \cos \omega t\} = 0 \tag{5.4}$$

for $a < t < b, -\infty < y < \infty$. For fixed t the left member of (5.4) is a polynomial of degree ≤ 3 in the variable y , vanishing for $-\infty < y < \infty$. Consequently, (5.4) holds if, and only if, the coefficients of this polynomial vanish for $a < t < b$.

(1) Let $Q = 0$. It follows that F [cf. (5.2)] is a connecting function if, and only if, $\gamma''(t) + \gamma(t) = (1 - \alpha)R \cos \omega t$ ($a < t < b$).

(2) Let $Q \neq 0$. Then F [cf. (5.2)] is a connecting function if, and only if, $\alpha = \pm 1$, $\gamma(t) \equiv 0$, $(\alpha - 1)R = 0$. Hence, for $R \neq 0$ we only have the trivial connecting function $F(t, y) = y$, while for $R = 0$ we also have the connecting function $F(t, y) = -y$.

5.3. Remarks

(A) There is some analogy between Theorem 1 (Section 3.1) and the Theorems in [7], [8] on the most general transformation that converts a given linear differential equation in an equation of the same type: for $n = 1$, $E = (a, b) \times R_1$, and the linear second order differential equation of the type considered in [8]; Theorem 1 follows from the theorem in [8].

(B) Let n be an integer ≥ 2 . Assume (2.2) is an equation which has been shown (by the methods of Chap. 4 or otherwise) to have no connecting function of order n . It then follows that (2.2) cannot be transformed into a linear equation by the method of [1].

(C) In [9] Miele used connecting functions of the form (4.14), (4.13) for solving (numerically) two-point boundary-value problems for linear nonhomogeneous differential equations.

The existence (and knowledge) of suitable connecting functions for nonlinear differential equations may lead (as in [9] for linear equations) to a method for solving nonlinear boundary value problems.

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