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ON THE RELATION BETWEEN STABILITY AND CONTRACTIVITY

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Dedicated to Professor Germund Dahlquist on the occasion of his sixtieth birthday.

Abstract.

This paper concerns the rate of growth of numerical approximations obtained by one-step methods for solving linear stiff initial value problems. For some of these methods weak stability with respect to arbitrary norms is shown to be equivalent to contractivity. This kind of stability is also proved to entail a barrier $p \leqq 1$ for the order of accuracy p within a broad class of methods, including general Runge-Kutta methods with $m \geqq 1$ stages.

1. Introduction.

This paper concerns the stability behaviour of numerical methods for solving stiff initial value problems in ordinary differential equations.

We shall focus on the behaviour of these methods when they are applied for approximating the solution $U(t)$ (for $t \leqslant 0$) to the (test) equation

$$\frac{d}{dt}U(t) = A(t)\,U(t) \tag{1.1}$$

when $U(0) = u_0$ is a given vector in the s-dimensional real vectorspace $\mathbb{R}^s$. We assume here that

(1.2a) $s \geqslant 1$, and $A(t)$ is a real $s \times s$ matrix depending continuously on $t \in \mathbb{R}$;

(1.2b) $|\cdot|$ is an arbitrary fixed norm on $\mathbb{R}^s$;

(1.2.c) each solution to (1.1) satisfies $|U(t)| \leqslant |U(t-h)|$ (for all $t \in \mathbb{R}$, $h > 0$).

We note that, under these assumptions, the given initial value problem for (1.1) can be arbitrarily stiff.

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In order to introduce the problem treated in this article we consider the familiar numerical method

$$u_n = u_{n-1} + hA(\tau_n)[\theta u_n + (1-\theta)u_{n-1}] \quad (n = 1, 2, 3, \ldots) \tag{1.3}$$

with $\tau_n = \theta t_n + (1-\theta)t_{n-1}$. Here $h > 0$ is the stepsize, $t_n = nh$ and $u_n \simeq U(t_n)$ $(n = 1, 2, 3, \ldots)$. Further θ is a real parameter specifying the method. Only for $\theta \geqslant \frac{1}{2}$ the method is A-stable (cf. [5]).

In [3] Brenner and Thomée presented a general theory which can be applied to (1.3) for the case where, in addition to (1.2), the matrix $A(t)$ is constant, i.e. $A(t) = A(0)$ (for all $t \in \mathbb{R}$). Under this additional assumption their theory implies, for $\theta \geqslant \frac{1}{2}$, that (1.3) is *weakly stable* in the sense that

$$|u_n| \leqslant \gamma n^q \cdot |u_0| \quad (\text{for all } n \geqslant 1,\ h > 0,\ u_0 \in \mathbb{R}^s) \tag{1.4}$$

for any sequence u_n satisfying (1.3). Here $q = \frac{1}{2}$ when $\theta = \frac{1}{2}$, and $q = 0$ when $\theta > \frac{1}{2}$. Further $\gamma < \infty$ depends on θ, but γ *does not depend on* $s \geqslant 1$ *or on* $A(0)$. It follows that the right-hand member in the inequality (1.4) does not depend on the stiffness of the given initial value problem. This fact is essential in our context of arbitrarily stiff problems, and can be exploited e.g. in the convergence analysis of "totally discrete" numerical methods for partial differential equations (cf. [2]).

The question arises whether (1.4) carries over to the general equation (1.1) where $A(t)$ is allowed to vary with $t \in \mathbb{R}$.

Finding answers to this question and to related ones is one of the purposes of the present paper. It turns out that the answer to the question above is negative. In fact we shall arrive at the surprising fact (see section 2) that a stability estimate of type (1.4) holds, uniformly for all equations (1.1) only in case for each individual equation (1.1) one has the estimate

$$|u_n| \leqslant |u_{n-1}| \quad (\text{for all } n \geqslant 1,\ h > 0,\ u_0 \in \mathbb{R}^s). \tag{1.5}$$

Clearly, property (1.5), which is called *contractivity* (cf. [7], [8]), implies (1.4) with $\gamma = 1$, $q = 0$ and is in general a much stronger property than (1.4). Relation (1.5) holds, for all problems (1.1), if and only if $\theta \geqslant 1$ (cf. [7]).

In section 2 we shall present theorem 1, which is basic for the present paper. The content of this theorem can be regarded as an extension of the above-mentioned equivalence between (1.4) and (1.5) to one-step methods that are more general than method (1.3).

In section 3 we shall formulate and prove a technical lemma which is essential for the proof of theorem 1 as given in section 2.

In section 4 we shall focus on the *order of accuracy* p of one-step methods for (1.1). Here we deal with a broad class of such methods, including e.g. general

Runge-Kutta schemes with $m \geqslant 1$ stages. It is known (cf. [8]) that if we require an arbitrary method out of this class to be *contractive* (cf. (1.5)) for all problems (1.1) satisfying (1.2), then the method is necessarily subject to the *order barrier* $p \leqslant 1$. A natural question is whether this barrier can be broken by relaxing the requirement of contractivity to the requirement of stability (in the sense of (1.4)).

Using theorem 1 we shall prove in section 4 that the answer to this question is negative. Theorem 4 of section 4 reveals that any method satisfying a stability estimate (similar to (1.4)) has necessarily an order $p \leqslant 1$.

REMARK 1. In condition (1.2) we allow arbitrary norms. If we would restrict our considerations to norms generated by an innerproduct in $\mathbb{R}^s$, the order barrier $p \leqslant 1$, mentioned above, would not hold (cf. [7]).

REMARK 2. Dealing with arbitrary norms has an obvious advantage over the restricted framework of innerproduct-norms. For instance,

$$A(t) = \begin{pmatrix} -1 & (1+|t|)^{-1} \\ 0 & 0 \end{pmatrix}$$

fulfils condition (1.2) (with the maximum norm), but would violate this condition if the norm in (1.2b) would be required to be generated by an innerproduct in $\mathbb{R}^2$. In the application to partial differential equations (see e.g. [2]) the advantage is even more pronounced as different norms on $\mathbb{R}^s$ need not be equivalent to each other *uniformly* for $s \geqslant 1$.

2. A theorem on weak stability and contractivity.

2.1. *Formulation of the theorem*

We first state some definitions and assumptions needed in the formulation of the subsequent theorem.

Suppose $|\cdot|$ is a norm on $\mathbb{R}^s$. Then for any $s \times s$ matrix A we define the *matrix norm* $\|A\|$ by

$$\|A\| = \sup\{|Ax| : x \in \mathbb{R}^s, |x| = 1\}$$

(cf. [6], [4]). Further the *logarithmic norm* $\mu[A]$ (cf. [4]), induced by $|\cdot|$, is defined by

$$\mu[A] = \lim_{h \to 0+} h^{-1}[\|I+hA\|-1].$$

In the following $\varphi(\zeta) = P(\zeta)/Q(\zeta)$ denotes a given rational function with polynomials $P(\zeta)$, $Q(\zeta) \not\equiv 0$ that have real coefficients and no common zeros.

For any $s \times s$ matrix A we say that $\varphi(A)$ *exists* and we write $\varphi(A) = P(A)Q(A)^{-1}$ whenever the matrix $Q(A)$ is invertible.

The function φ is called *absolutely monotonic* on the interval $J \subset \mathbb{R}$ if J contains no singularities of φ and $\varphi(\zeta)$ as well as all derivatives $\varphi^{(j)}(\zeta)$, $(j = 1, 2, 3, \ldots)$ are nonnegative for all $\zeta \in J$.

THEOREM 1. *Let $\varphi(\zeta)$ be a rational function with $\varphi(0) = 1$. Then the following statements (i), (ii), (iii) are equivalent.*

(*i*) *There exist constants $\gamma < \infty$, $q < \infty$ (depending only on φ) such that*

$$\|\varphi(A_n)\varphi(A_{n-1})\cdot\ldots\cdot\varphi(A_1)\| \leqslant \gamma \cdot n^q$$

whenever $n \geqslant 1$, $s \geqslant 1$, $|\cdot|$ is any norm on $\mathbb{R}^s$ and A_j are $s \times s$ matrices with $\mu[A_j] \leqslant 0$ such that $\varphi(A_j)$ exist $(j = 1, 2, \ldots, n)$. Here $\|\cdot\|$ and $\mu[\cdot]$ denote the matrix norm and logarithmic norm induced by the norm $|\cdot|$ in $\mathbb{R}^s$.

(ii) *The matrix $\varphi(A)$ exists and satisfies the inequality $\|\varphi(A)\| \leqslant 1$ whenever $s \geqslant 1$, $|\cdot|$ is any norm on $\mathbb{R}^s$ and A is an $s \times s$ matrix with $\mu[A] \leqslant 0$. Here $\|\cdot\|$, $\mu[\cdot]$ have the same meaning as in statement (i).*

(iii) *The function $\varphi(\zeta)$ is absolutely monotonic on $(-\infty, 0]$.*

2.2. *Explanation ot the theorem*

In order to interpret theorem 1 consider a sequence of vectors $u_n \in \mathbb{R}^s$ satisfying the recurrence relation

$$u_n = \varphi(A_n)\, u_{n-1} \quad (n = 1, 2, 3, \ldots), \tag{2.1}$$

with φ as in the theorem above. Suppose the u_n satisfy a *weak stability* estimate of type (1.4) whenever $s \geqslant 1$, $|\cdot|$ is any norm in $\mathbb{R}^s$ and $\mu[A_n] \leqslant 0$ $(n \geqslant 1)$. Then theorem 1 allows us to conclude that, for all such A_n, the vectors u_n satisfy in fact a (stronger) *contractivity* estimate of type (1.5).

An example of (2.1), with $\varphi(\zeta) = (1+(1-\theta)\zeta)(1-\theta\zeta)^{-1}$, is provided by method (1.3). Here $A_n = hA(\tau_n)$, and when $A(t)$ satisfies assumption (1.2) we can conclude, by the subsequent lemma, that $\mu[A_n] \leqslant 0$.

LEMMA 2. *Let $A(t)$ satisfy (1.2a), and assume (1.2b). Then (1.2c) holds if and only if*

$$\mu[A(t)] \leqslant 0 \qquad (t \in \mathbb{R}). \tag{2.2}$$

Here $\mu[\cdot]$ denotes the logarithmic norm induced by the norm occurring in assumption (1.2b).

PROOF. If (2.2) holds, then we have (1.2.c) by [4, p. 14]. Conversely assume (1.2c). Let $\tau \in \mathbb{R}$, $x \in \mathbb{R}^s$, $|x| = 1$, $h > 0$ and define

$$f(h,x) = h^{-1}[|x+hA(\tau)x|-|x|].$$

We have
$$\mu[A(\tau)] = \lim_{h\to 0+} \sup_{|x|=1} f(h,x).$$

Let $U(t)$ satisfy (1.1) and $U(\tau) = x$. Defining $\varepsilon(x,h)$ by the relation $|x+hA(\tau)x| = |U(\tau+h)|+h\varepsilon(x,h)$, and using (1.2.c), we obtain

We have
$$f(h,x) \leqslant \varepsilon(h,x).$$

It can be proved that $\lim_{h\to 0+} \sup_{|x|=1} f(h,x) \leqslant 0,$

e.g. by noting that $\varepsilon(h,x) \to 0$ (as $h \to 0+$) uniformly for $|x| = 1$. This implies $\mu[A(\tau)] \leqslant 0$ and completes the proof. ■

2.3. *Proof of theorem* 1.

Assume statement (iii) holds. Then statement (ii) holds in view of [8, theorem 2.4.].

Assume statement (ii) holds. Then (i) holds as well with $\gamma = 1$, $q = 0$.

Assume statement (i) holds. We shall complete the proof of the theorem by deducing (iii).

Let $\zeta < 0$ be given. By considering the case $s = 1$, we see that $\varphi(\zeta)$ must be regular at $\zeta = \xi$. Suppose, contrary to (iii), that $\varphi^{(m)}(\xi) < 0$ for some $m \geqslant 0$.

Defining $\gamma_j = (j!)^{-1}\varphi^{(j)}(\xi)(-\xi)^j$ (for $j \geqslant 0$) there follows $\gamma_m < 0$. Since $\varphi(0) = 1$ we have

$$|\gamma_0|+|\gamma_1|+\ldots+|\gamma_p| > 1$$

for p sufficiently large. Let n be an arbitrary integer $\geqslant 1$ and let $\|\cdot\|_\infty, \mu_\infty[\cdot]$ be as in section 3. Applying lemma 4 (see section 3) we obtain

$$\|\varphi(A_n)\cdot\varphi(A_{n-1})\cdot\ldots\cdot\varphi(A_1)\|_\infty \geqslant \lambda^n$$

with $\lambda = |\gamma_0|+|\gamma_1|+\ldots+|\gamma_p|$ and $s \times s$ matrices A_j satisfying $\mu_\infty[A_j] \leqslant 0$. By (i) we obtain $\gamma n^q \geqslant \lambda^n$ ($n = 1, 2, 3, \ldots$), which is impossible. ■

3. Formulation and proof of lemma 3.

The purpose of this section is to state and prove the lemma which played a key-role in the proof of theorem 1 above.

With $|\cdot|_\infty$ we denote the maximum norm in $\mathbb{R}^s$, and with $\|\cdot\|_\infty$, $\mu_\infty[\cdot]$ we denote the matrix norm and logarithmic norm induced by $|\cdot|_\infty$. Explicit expressions for $\|A\|_\infty$, $\mu_\infty[A]$ in terms of the entries of the matrix A can be found in [4, p. 11]. With I we denote the identity matrix.

LEMMA 3. *Let ξ be a given real number and let the rational function $\varphi(\zeta)$ be regular at $\zeta = \xi$. Let $\gamma_j = (j!)^{-1}\ \varphi^{(j)}(\xi)(-\xi)^j$ for $j = 0, 1, 2, \ldots$ Assume $n \geqslant 1$, $p \geqslant 1$ are any given integers.*

Then there exist an integer $s \geqslant 1$ and $s\times s$ matrices A_j such that

$$\|A_j - \xi I\|_\infty \leqslant |\xi| \qquad (j = 1, 2, \ldots, n), \tag{3.1.a}$$

$$\varphi(A_j) \text{ exist} \qquad (j = 1, 2, \ldots, n), \tag{3.1.b}$$

$$\|\varphi(A_n)\varphi(A_{n-1})\cdot\ldots\cdot\varphi(A_1)\|_\infty \geqslant (|\gamma_0|+|\gamma_1|+\ldots+|\gamma_p|)^n. \tag{3.1.c}$$

PROOF. 1. Let ξ, φ, γ_j, n, p be as in the lemma above.

For any $s\times s$ matrix B with spectrum $\sigma(B) = \{0\}$, the matrix $\varphi(\xi I - \xi B)$ exists and $\varphi(\xi I - \xi B) = \gamma_0 + \gamma_1 B + \gamma_2 B^2 + \ldots$ (see [6, p. 568]).

Let B_j be any $s\times s$ matrices with $\sigma(B_j) = \{0\}$ and $\|B_j\|_\infty \leqslant 1$ $(j = 1, 2, \ldots, n)$. We define $A_j = \xi I - \xi B_j$ and $P_n = \varphi(A_n)\varphi(A_{n-1})\cdot\ldots\cdot\varphi(A_1)$. Clearly (3.1.a), (3.1.b) are fulfilled, so that it only remains to show that s and B_j can be chosen such that, additionally,

$$\|P_n\|_\infty \geqslant (|\gamma_0|+|\gamma_1|+\ldots+|\gamma_p|)^n. \tag{3.2}$$

Writing $Q_j = \gamma_0 + \gamma_1 B_j + \ldots + \gamma_p (B_j)^p$, $R_j = \gamma_{p+1}(B_j)^{p+1} + \gamma_{p+2}(B_j)^{p+2} + \ldots$, we have $P_n = (Q_n + R_n)(Q_{n-1}+R_{n-1})\cdot\ldots\cdot(Q_1+R_1)$ and therefore

$$P_n = S_n + T_n \tag{3.3}$$

with $S_n = Q_n Q_{n-1}\cdot\ldots\cdot Q_1$, $T_n = (Q_n+R_n)\cdot\ldots\cdot(Q_1+R_1) - Q_n\cdot\ldots\cdot Q_1$.

We choose $B_j = E^{\beta(j)}$ where E is the $s\times s$ matrix all of whose entries E_{qr} vanish with the exception of $E_{q,q+1} = 1$ $(1 \leqslant q \leqslant s-1)$. Further $b = (\beta(1), \beta(2), \ldots, \beta(n))$ is a vector of positive integers $\beta(j)$ to be specified below $(j = 1, 2, \ldots, n)$. We easily obtain the expressions

$$S_n = \sum_{x\in\Gamma} \gamma_{\xi(1)}\cdot\ldots\cdot\gamma_{\xi(n)} E^{(b,x)},$$

$$T_n = \sum_{y\in\Delta} \gamma_{\eta(1)}\cdot\ldots\cdot\gamma_{\eta(n)} E^{(b,y)},$$

where we use the notations $x = (\xi(1), \xi(2), \ldots, \xi(n))$, $y = (\eta(1), \eta(2), \ldots, \eta(n))$ and

$(b, x) = \beta(1)\xi(1)+\ldots+\beta(n)\xi(n)$, $(b, y) = \beta(1)\eta(1)+\ldots+\beta(n)\eta(n)$. With Γ we have denoted the set of all vectors x with integer components $\xi(j)$, $0 \leqslant \xi(j) \leqslant p$, and with Δ we have denoted the set of all $y \notin \Gamma$ with integer components $\eta(j) \geqslant 0$ $(j = 1, 2, \ldots, n)$.

Suppose s and b are such that

(3.4.a) $$(b, x) \neq (b, x') \qquad \text{(for all } x \neq x' \text{ in } \Gamma\text{)},$$

(3.4.b) $$(b, x) \neq (b, y) \qquad \text{(for all } x \in \Gamma,\ y \in \Delta\text{)},$$

(3.4.c) $$\max\{(b, x) | x \in \Gamma\} < s.$$

Then we obtain from (3.3) and the expressions above for S_n, T_n the lower bound $\|P_n\|_\infty \geqslant \Sigma|\gamma_{\xi(1)} \cdot \ldots \cdot \gamma_{\xi(n)}|$ where the summation is for all $x \in \Gamma$. This implies (3.2).

We shall complete the proof of lemma 3 by showing below that s, b exist satisfying (3.4).

2. For any vector $b = (\beta(1), \beta(2), \ldots, \beta(n))$ with positive integer components one has (for $x \in \Gamma$ and $y \in \Delta$)

$$(b, x) \leqslant n|x|_\infty \cdot \max_j \beta(j), \qquad |y|_\infty \cdot \min_j \beta(j) \leqslant (b, y).$$

Suppose b is such that

(3.5.a) $$\max_j \beta(j) \leqslant 2 \min_j \beta(j).$$

We then obtain, in view of $|x|_\infty \leqslant p$, the implication

(3.6.a) $$[x \in \Gamma,\ y \in \Delta,\ |y|_\infty > 2np] \Rightarrow (b, x) < (b, y).$$

We define Ω to be the set of all vectors $z = (\zeta(1), \zeta(2), \ldots, \zeta(n))$ with integer components such that $0 < |z|_\infty \leqslant 2np$. Suppose b is such that

(3.5.b) $$(b, z) \neq 0 \qquad \text{(for all } z \in \Omega\text{)}.$$

Then we obtain the implication

(3.6.b) $$[x \in \Gamma,\ y \in \Delta,\ |y|_\infty \leqslant 2np] \Rightarrow (b, x) \neq (b, y),$$

as well as relation (3.4.a).

Since (3.6.a, 3.6.b) imply (3.4.b) it is sufficient to show that there exist s, b satisfying (3.5.a), (3.5.b), (3.4.c).

We try to find a suitable $b = (\beta(1), \beta(2),\ldots,\beta(n))$ of the form

(3.7) $$\beta(j) = N \cdot (1+N^{-1}\theta(j)) \qquad (j = 1, 2,\ldots, n)$$

where N, $\theta(j)$ are integers and $0 < \theta(j) < N$. Clearly (3.5.a), (3.5.b) are fulfilled whenever

(3.8) $$\sum_{j=1}^{n} \zeta(j) \cdot (1+N^{-1}\theta(j)) \neq 0 \qquad \text{(for all } z \in \Omega\text{)}.$$

Since there exists a real $t \in (0, 1)$ with

$$\zeta(1)(1+t)+\zeta(2)(1+t^2)+\ldots+\zeta(n)(1+t^n) \neq 0$$

(uniformly for all $z \in \Omega$) we can choose N, $\theta(j)$ such that (3.8) holds. Defining $\beta(j)$ by (3.7) and s by $s = n \cdot 2Np$, the requirements (3.5.a), (3.5.b), (3.4.c) are thus fulfilled. ■

4. An order barrier for weakly stable methods.

4.1. *Formulation of the order barrier*

In this section we deal with variable stepsizes $h_n > 0$ and gridpoints $t_0 = 0$, $t_n = h_1+h_2+\ldots+h_n$ $(n \geq 1)$. We consider methods producing approximations u_n to $U(t_n) \in \mathbb{R}^s$ that can be written in the general form

(4.1) $$u_n = \Phi(h_nA(\tau_{n,1}), h_nA(\tau_{n,2}),\ldots,h_nA(\tau_{n,m}))u_{n-1} \qquad (n = 1, 2, 3,\ldots).$$

Here $m \geq 1$, $\tau_{n,i} = t_{n-1}+c_ih_n$ $(i = 1, 2,\ldots,m)$, where $c_1, c_2,\ldots,c_m$ are coefficients which, together with the function Φ, specifiy the method.

$\Phi(A_1, A_2,\ldots,A_m)$ stands for an $s \times s$ matrix whenever $A_1, A_2,\ldots,A_m$ are $s \times s$ matrices for which $\Phi(A_1, A_2,\ldots,A_m)$ is defined. In view of (1.2) and lemma 2 we make the natural assumption that the function Φ has a domain of definition $\mathscr{D}$ which is contained in $\mathscr{E}$. Here $\mathscr{E}$ denotes the collection of all sequences $(A_1, A_2,\ldots,A_m)$ composed of square matrices A_i, of an order $s \geq 1$, for which a norm $|\cdot|$ in $\mathbb{R}^s$ exists which induces simultaneously $\mu[A_i] \leq 0$ $(i = 1, 2,\ldots,m)$.

We assume that

(4.2.a) for each $(A_1,\ldots,A_m) \in \mathscr{E}$ we have

$$(hA_1,\ldots,hA_m) \in \mathscr{D} \text{ with } \Phi(hA_1,\ldots,hA_m) \to I \text{ when } h \to 0+.$$

We also assume that a rational function $\varphi(\zeta)$ exists such that

(4.2.b) if $(A, A,\ldots,A) \in \mathscr{E}$ and $\varphi(A)$ exists, then

$$(A, A,\ldots,A) \in \mathscr{D} \text{ with } \Phi(A, A,\ldots,A) = \varphi(A).$$

One easily verifies that e.g. general Runge-Kutta methods and Rosenbrock methods are of type (4.1) with a Φ fulfilling the assumptions above (cf. also [8, section 2.3.3]).

We recall that the *order* p of method (4.1) is the largest integer p for which $U(t_n)$ satisfies (4.1) up to an error $= O((h_n)^{p+1})$ (for $h_n \to 0+$) whenever $U(t), A(t)$ satisfy (1.1) and are sufficiently differentiable.

We say that method (4.1) is *weakly stable* if there exist $\gamma < \infty$, $q < \infty$ such that

$$|u_n| \leqslant \gamma n^q |u_0|$$

whenever $n \geqslant 1$, $h_n > 0$, (1.2), $u_0 \in \mathbb{R}^s$ and u_n satisfies (4.1).

THEOREM 4. *Let method* (4.1) *satisfy assumption* (4.2.). *If the method is weakly stable, then its order p cannot be greater than one.*

4.2. *Proof of theorem* 4.

1. Assume (4.2) and weak stability for method (4.1.).

From (4.2) we deduce that $\varphi(\zeta)$ is regular at $\zeta = 0$ with $\varphi(0) = 1$. Suppose statement (i) (of theorem 1) holds. By theorem 1 we then conclude that statement (iii) holds as well. The absolute monotonicity of φ implies that for $p > 1$, the relation $\varphi(\zeta) - \exp(\zeta) = O(\zeta^{p+1})$ (for $\zeta \to 0$) is impossible (cf. [1] or [8]). In view of (4.2.b) it follows that the order p of (4.1) satisfies $p \leqslant 1$.

It remains to show that statement (i) holds.

2. Let $n \geqslant 1$, $s \geqslant 1$, $|\cdot|$, A_j be as in (i). Let $\varepsilon > 0$ be given.

We shall prove below, in part 3, that there exist $h_j > 0$, $A(t)$ such that, with the notation

$$H_j = (h_j A(\tau_{j,1}), h_j A(\tau_{j,2}),\ldots,h_j A(\tau_{j,m})),$$

we have for $1 \leqslant k \leqslant n-1$

(4.3.a) $$H_{2k} \in \mathscr{D} \text{ and } \|\Phi(H_{2k}) - I\| \leqslant \varepsilon,$$

and for $0 \leqslant k \leqslant n-1$

(4.3.b) $$H_{2k+1} \in \mathscr{D} \text{ and } \Phi(H_{2k+1}) = \varphi(A_{k+1}),$$

while

(4.4) $A(t)$ satisfies (1.2) with the above s and $|\cdot|$.

In view of the weak stability this implies

$$\|\Phi(H_{2n-1})\cdot\ldots\cdot\Phi(H_2)\Phi(H_1)\| \leqslant \gamma(2n-1)^q.$$

Defining $E_k = \Phi(H_{2k})-I$ this inequality can be written in the form

$$\|\varphi(A_n)(I+E_{n-1})\varphi(A_{n-1})\cdot\ldots\cdot(I+E_1)\varphi(A_1)\| \leqslant \gamma(2n-1)^q.$$

In view of (4.3.a) we have $\|E_k\| \leqslant \varepsilon$, and therefore

$$\|\varphi(A_n)\cdot\varphi(A_{n-1})\cdot\ldots\cdot\varphi(A_1)\| \leqslant \gamma 2^q\cdot n^q + \sum_{r=1}^{n-1}\left(\prod_{k=1}^{n}\|\varphi(A_k)\|\right)\binom{n-1}{r}\varepsilon^r.$$

By letting $\varepsilon \to 0$ we see that (i) holds.

3. In the subsequent proof of (4.3), (4.4) we use the notation

$$T_j = \{\tau_{N,i} | 1 \leqslant N \leqslant j,\ 1 \leqslant i \leqslant m\}.$$

We define $h_1 = 1$, $A(\tau_{1,i}) = A_1$ $(1 \leqslant i \leqslant m)$, and we shall define recursively h_j, $A(\tau_{j,i})$ for $1 < j \leqslant 2n-1$ such that

(4.5) $$\mu[A(\tau_{j,i})] \leqslant 0 \qquad (1 \leqslant i \leqslant m).$$

Note that, in view of the assumption in (i) on A_j, (4.5) holds for $j = 1$ and (4.3.b) holds for $k = 0$.

Let $1 \leqslant k \leqslant n-1$, and suppose h_j, $A(\tau_{j,i})$ are already defined, such that (4.5) holds, for $1 \leqslant j \leqslant 2k-1$.

We define $h_{2k} = \alpha$, $h_{2k+1} = 1$ where $\alpha > 0$ is so small that

$$t_{2k-1}+\alpha+c_i \notin T_{2k-1}\ (1 \leqslant i \leqslant m).$$

We further define $A(\tau_{2k,i}) = A_{k+1}$ (for $\tau_{2k,i} \in T_{2k}-T_{2k-1}$) and $A(\tau_{2k+1,i}) = A_{k+1}$ (for $\tau_{2k+1,i} \in T_{2k+1}-T_{2k}$).

This definition implies (4.5) for $1 \leqslant j \leqslant 2(k+1)-1$.

Further, since $\tau_{2k+1,i} \notin T_{2k-1}$ $(1 \leqslant i \leqslant m)$, there follows $A(\tau_{2k+1,i}) = A_{k+1}$ $(1 \leqslant i \leqslant m)$. In view of (4.2.b) our definition thus implies (4.3.b).

Writing $H(\alpha) = (\alpha A(\tau_{2k,1}), \alpha A(\tau_{2k,2}),\ldots,\alpha A(\tau_{2k,m}))$ and using (4.2.a) it follows that

$$H(\alpha) \in \mathscr{D} \quad \text{and} \quad \|\Phi(H(\alpha)) - I\| \leqslant \varepsilon,$$

when we choose $\alpha > 0$ sufficiently small. Our definition thus also implies (4.3.a).

Applying the definition above successively for $k = 1, 2,\ldots,n-1$ we obtain h_j, $A(\tau_{j,i})$ which fulfil (4.5) (for $1 \leqslant j \leqslant 2n-1$), (4.3.b) (for $0 \leqslant k \leqslant n-1$) and (4.3.a) (for $1 \leqslant k \leqslant n-1$).

We finally define $A(t)$ for $t \notin T_{2n-1}$ by linear interpolation and by $A(t) = A(\min T_{2n-1})$ (for $t < \min T_{2n-1}$), $A(t) = A(\max T_{2n-1})$ (for $t > \max T_{2n-1}$). Since (4.5) holds for all $\tau_{j,i} \in T_{2n-1}$ we have $\mu[A(t)] \leqslant 0$ (for all $t \in \mathbb{R}$), and by lemma 2 also (4.4). This completes the proof. ■

REMARK. Suppose method (4.1) has the property that $c_i - c_j$ is not an integer (for $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m, i \neq j$). Then theorem 4 would remain valid if the requirement of weak stability would be relaxed in that only *fixed stepsizes* $h_n = h > 0$ ($n = 1, 2, 3,\ldots$) are considered. The proof of this modified version of theorem 4 is even simpler than our proof above. However, this simpler proof does not carry over if the c_i do not have the property above.

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