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STABILITY AND BOUNDEDNESS IN THE NUMERICAL SOLUTION OF INITIAL VALUE PROBLEMS

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Abstract. This paper concerns the theoretical analysis of step-by-step methods for solving initial value problems in ordinary and partial differential equations.

The main theorem of the paper answers a natural question arising in the *linear stability analysis* of such methods. It guarantees a (strong) version of numerical stability—under a stepsize restriction related to the stability region of the numerical method and to a circle condition on the differential equation.

The theorem also settles an open question related to the properties *total-variation-diminishing*, *strong-stability-preserving*, *monotonic* and *(total-variation-)bounded*. Under a monotonicity condition on the forward Euler method, the theorem specifies a stepsize condition guaranteeing boundedness for linear problems.

The main theorem is illustrated by applying it to linear multistep methods. For important classes of these methods, conclusions are thus obtained which supplement earlier results in the literature.

1. Introduction

We shall address various related questions arising in the numerical solution of initial value problems. In Sections 1.1, 1.2 of this introduction, these questions will be formulated and put in an historical context. In Section 1.3, we shall give an outline of the rest of the paper.

1.1. Numerical stability.

Bounding the difference between two series of approximations. Below, we shall denote by $\mathbb{V}$ an arbitrary real or complex vectorspace $\mathbb{V}$, with seminorm $||v||$ for $v \in \mathbb{V}$.[1] Consider an initial value problem in $\mathbb{V}$ that can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\,t}U(t) = F(U(t)) \ \ (\text{for } t > 0), \quad U(0) = u_0, \tag{1.1}$$

where $F : \mathbb{V} \to \mathbb{V}$ and $u_0 \in \mathbb{V}$ are given, whereas $U(t) \in \mathbb{V}$ is unknown for $t > 0$.

Current numerical methods for solving (1.1) generate, in a step-by step fashion, approximations u_n of $U(t)$ at consecutive grid-points $t = t_n$. An essential requisite of the methods is numerical *stability*—in the sense that any (discretization- or rounding-)errors, introduced at some stage of the computations, are propagated "mildly", in the subsequent computations. For this kind of stability, it is essential

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[1]This means: $||\lambda \cdot v|| = |\lambda| \cdot ||v||$ and $||v + w|| \leq ||v|| + ||w||$ for all scalars λ and $v, w \in \mathbb{V}$.

that the *difference* $\tilde{u}_n - u_n$ between two approximations u_n and $\tilde{u}_n$, does *not* grow "fast" (as n increases); cf. e.g. [8] (sections II.3, III.4), [37] (section 4).

To be more specific, we consider the general *linear multistep method* (LMM); see e.g. [2], [8], [9]. The method, applied to problem (1.1), yields approximations u_n (for $n \geq k$), with

$$u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k} + \Delta t \left[b_0 F(u_n) + \cdots + b_k F(u_{n-k}) \right]. \tag{1.2}$$

Here $\Delta t > 0$ denotes the stepsize, and $u_n \approx U(t_n)$, with $t_n = n\,\Delta t$; further, $k \geq 1$ is a fixed integer, and a_j, b_j are real coefficients specifying the LMM, with

$$\sum_{j=1}^{k} a_j = 1, \qquad \sum_{j=1}^{k} j\, a_j = \sum_{j=0}^{k} b_j. \tag{1.3}$$

Special attention to stability of these methods was paid, in the literature, for the case where the differential equation stands for a linear partial differential equation, or a semidiscrete (method of lines) version thereof. Consider problem

$$\tfrac{\mathrm{d}}{\mathrm{d}\,t} U(t) = L\,U(t) \ \ (\text{for } t > 0), \quad U(0) = u_0, \tag{1.4}$$

where $L : \mathbb{V} \to \mathbb{V}$ is a linear operator; in this case the LMM formula takes on the form

$$u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k} + \Delta t \left[b_0 L(u_n) + \cdots + b_k L(u_{n-k}) \right]. \tag{1.5}$$

Because of linearity, the difference $\tilde{u}_n - u_n$ between two sequences $\tilde{u}_n$ and u_n, obtained via the formula, satisfies still (1.5); for numerical stability, in this case, it is thus crucial to have moderate bounds on $||u_N||$ (for $N \geq k$) as soon as (1.5) holds for $k \leq n \leq N$. In the literature, such bounds were established, indeed, notably of the form

$$||u_N|| \leq \mu\, N^{\alpha} \cdot \max_{0 \leq j \leq k-1} ||u_j|| \quad (\text{whenever } u_n \text{ satisfies (1.5) for } k \leq n \leq N). \tag{1.6}$$

Here μ, α stand for non-negative constants that are of moderate size and independent of critical variables, such as $N \geq k$ and $\Delta t \to 0$. The case where $\alpha = 0$, is of course preferred; it is related to the form of stability occurring in the Lax equivalence theorem; cf. e.g. [30].

The stability region of linear multistep methods. Consider the LMM in the test situation $\mathbb{V} = \mathbb{C}$. Putting $z = \Delta t\, L$, formula (1.5) (with $n \geq k$) now reduces to a scalar recurrence relation with characteristic polynomial

$$P(z, \lambda) = (1 - b_0 z)\lambda^k - (a_1 + b_1 z)\lambda^{k-1} - \cdots - (a_k + b_k z).$$

We will say that any polynomial (with complex coefficients) satisfies the *root condition*, if its roots λ have a modulus $|\lambda| \leq 1$, while roots with $|\lambda| = 1$, are simple. The well-known *stability region* of the LMM, denoted by S, can be defined as the set of all $z \in \mathbb{C}$ with $1 - b_0 z \neq 0$, for which $P(z, \lambda)$ (as a polynomial in the variable λ) satisfies the root condition.

The stability region is a standard tool for getting insight into the stability behaviour of LMMs. But, the region is essentially defined in terms of the method's behaviour when applied to a very simple (scalar) test problem. Hence, in case of more general (non-scalar) problems of type (1.4), the region S should be used carefully, to avoid stability conclusions that are, in reality, false; cf. e.g. [4] (section 1.3), [7], [19] (section 4), [24], [28], [29], [36].

In order to arrive at correct conclusions, by using stability regions, basic assumptions on the operator $\Delta t\, L$ should be made that are stronger than a mere premise about its eigenvalues, or spectrum. Such stronger assumptions, and corresponding estimates of type (1.6), were dealt with in the literature; see e.g. [13], [23], [26], [27], [29].

We note that—due to linearity of L—the stability estimates in the last references are also relevant to solving non-homogeneous equations $\frac{\mathrm{d}}{\mathrm{d}\,t}U(t) = L\,U(t) + f(t)$. Moreover, they are relevant for cases to which classical Fourier transformations do not apply, e.g., when irregular grids are involved or spectral methods are used; and they are not limited to seminorms generated by (semi-)inner products, so that, e.g., the maximum-norm is included.

The present paper will deal with stability estimates, relevant to cases mentioned in the last paragraph, under the well-known assumption that $\Delta t\, L$ satisfies a *circle condition*:

(1.7) $$||(\Delta t\, L + \gamma I)\, v|| \leq \gamma\, ||v|| \text{ (for all } v \in \mathbb{V}).$$

Here $\gamma > 0$, and I denotes the identity operator in $\mathbb{V}$. Condition (1.7) was used earlier in the analysis of numerical methods; cf. e.g. [9] (section IV.11), [18] (section 3), [19], [26], [36], and the references therein. It implies in general that the eigenvalues of $\Delta t\, L$ are situated within, or on, the circle in the complex plane with center $z = -\gamma$ and radius γ. But, conversely, this property of the eigenvalues is in general *not* strong enough to imply (1.7).

For a restricted class of LMMs and under conditions which do *not* follow from the circle condition, neat estimates of type (1.6) were derived, in the literature, with $\alpha = 0$. But, as far as the author knows, estimates with $\alpha = 0$ and relevant to general LMMs under the circle condition, are lacking in the literature. The question thus poses itself of whether this gap in existing literature can be filled up. An analogous question poses itself for multistage versions of LMMs. These questions will be addressed in the present paper; cf. Section 1.3.

1.2. **Monotonicity and boundedness.**

Monotonicity. Questions related to those just mentioned, occur in the study of the special properties *total-variation-diminishing*, *strong-stability-preserving*, *monotonicity* and *(total-variation) boundedness*; cf. e.g. [6], [12], [15], [16], [33], [34], [38]. We shall shortly review some of these properties, using the same notations and assumptions as above.[2]

The last publications start generally by assuming that, for a specific constant $\tau > 0$,

(1.8) $$||v_0 + \tau_0 F(v_0)|| \leq ||v_0|| \quad \text{(for any } \tau_0 \text{ with } 0 < \tau_0 \leq \tau, \text{ and any } v_0 \in \mathbb{V}).$$

Under this assumption, the LMM (1.2) has been considered with stepsize Δt restricted by

(1.9) $$0 < \Delta t \leq \gamma \cdot \tau,$$

where the coefficient $\gamma > 0$ only depends on the coefficients a_j, b_j of the LMM. Special LMMs and corresponding γ were determined, such that (1.8), (1.9) imply

[2]In some of the listed papers, it is assumed that $||\cdot||$ is an arbitrary convex functional; for simplicity, we shall still assume that $||\cdot||$ is a seminorm.

(for all $N \geq k$):

(1.10)
$$||u_N|| \leq \max_{0 \leq j \leq k-1} ||u_j|| \quad \text{(when } u_N \text{ is generated by the LMM from } u_0, \ldots, u_{k-1}\text{)}.$$

Property (1.10) is often referred to as *monotonicity* or *strong stability*; it is of particular importance in the numerical solution of initial value problems arising by semi-discretization (method of lines) of time dependent partial differential equations. An important choice for $||\cdot||$, occurring in that context, is the *total variation seminorm* $||v|| = ||v||_{TV} = \sum_i |v^{(i)} - v^{(i-1)}|$ (for vectors v with components $v^{(i)}$). Processes that are monotonic with regard to that seminorm, play a special role in the solution of hyperbolic conservation laws and are called *total-variation-diminishing* (TVD); cf. e.g. [6], [10], [17], [22], [33], [34].

Boundedness. Total-variation-diminishing processes are trivially *total-variation-bounded* in the sense that a constant μ exists (independent of critical variables, like $N \geq k$ and $\Delta t \to 0$) such that

$$||u_N||_{TV} \leq \mu \cdot \max_{0 \leq j \leq k-1} ||u_j||_{TV} \text{ (when } u_N \text{ is generated by the LMM from } u_0, \ldots, u_{k-1}\text{)}.$$

In the solution of hyperbolic conservation laws, this property is crucial for suitable convergence properties when $\Delta t \to 0$; see e.g. [22], [17]. That is one of the underlying reasons why attention has been paid in the literature to the monotonicity property (1.10).

Unfortunately, for many important LMMs—including all Adams methods and backward differentiation methods, with $k > 1$—*there exists no stepsize-coefficient* $\gamma > 0$ such that (1.8), (1.9) imply monotonicity in the sense of (1.10); see e.g. [15], [20], [39].

Accordingly, along with monotonicity, also directly the weaker *boundedness property*

(1.11)
$$||u_N|| \leq \mu \cdot \max_{0 \leq j \leq k-1} ||u_j|| \quad \text{(when } u_N \text{ is generated by the LMM from } u_0, \ldots, u_{k-1}\text{)}$$

has been studied, where μ is possibly greater than 1 (but still independent of critical variables, like $N \geq k$ and $\Delta t \to 0$). Conditions on γ were given such that this boundedness property holds under conditions (1.8), (1.9); see [14], [15], [16], [32].

Although monotonicity and boundedness were primarily considered with a view to solving non-linear hyperbolic problems, it is worthwhile to study these properties especially for linear problems (1.4) as well; see e.g. [6] (chapter 4), [12] (section 3). In solving (1.4), property (1.11) just amounts to (1.6) with $\alpha = 0$, and assumption (1.8) then reduces to

$$||v_0 + \tau_0 \, L \, v_0|| \leq ||v_0|| \quad \text{(for any } \tau_0 \text{ with } 0 < \tau_0 \leq \tau, \text{ and any } v_0 \in \mathbb{V}\text{)}. \tag{1.12}$$

In the context of solving just problems of type (1.4), there are still important LMMs for which *no* $\gamma > 0$ *exists* such that (1.12), (1.9) imply monotonicity; cf. [35] (p. 283), [20], [21]. Moreover, the conditions on γ, just mentioned and relevant to (1.11), were obtained in the context of general (non-linear) problems (1.1), and they are far from simple.

The natural question thus arises of whether, just for problems of type (1.4), less restrictive and more simple conditions on γ exist such that (1.12), (1.9) imply the boundedness property (1.11). An analogous question poses itself for multistage versions of LMMs. In the present paper, we shall also address these questions; cf. Section 1.3.

1.3. **Outline of the rest of the paper.** In Section 2, we shall first introduce a general class of multistage multistep methods which encompasses LMMs and is relevant to problem (1.4). Next, our main result, Theorem 2.1, will be formulated. It guarantees, for all methods of the general class, an extended version of property (1.6) with $\alpha = 0$, under a suitable circle condition on $\Delta t\, L$. It specifies also a stepsize-coefficient γ such that conditions (1.12), (1.9) imply an extended version of the boundedness property (1.11) (with regard to (1.4)). The theorem is best possible in a sense specified at the end of Section 2, and it settles essentially the questions (pertinent to multistage versions of LMMs) raised at the end of Sections 1.1 and 1.2.

In Section 3, the general theory will be applied to LMMs. Theorem 3.1 resolves explicitly the questions about LMMs raised at the end of Sections 1.1 and 1.2. Moreover, Corollary 3.3 gives a neat criterion for the existence of $\gamma > 0$, such that conditions (1.12), (1.9) imply the boundedness property (1.11) (with regard to (1.4)). Next, for classes of important LMMs, conclusions are obtained, via Corollary 3.3, supplementing earlier results in the literature.

Sections 4 and 5 concern the proof of Theorem 2.1.

Section 4 is about the relation between circle condition (1.7) and conditions (1.12), (1.9). The section studies also the relation between, on the one hand, the setting considered so far of arbitrary vectorspaces $\mathbb{V}$ with seminorm $\|\cdot\|$, and, on the other hand, the (a priori) more restricted setting of Banach spaces $\mathbb{V}$ with norm $\|\cdot\|$.

Because of the relations considered in Section 4, the actual proof of Theorem 2.1, in Section 5, comes down to proving a boundedness estimate, under condition (1.7), in a Banach space setting.

2. Formulation of the main result of the paper

We shall study a generic numerical process, relevant to problem (1.4), using the notations and assumptions of Section 1. The process consists in computing, for $n \geq 1$, numerical approximations $w_{n1}, w_{n2}, \ldots, w_{nk} \in \mathbb{V}$, with

(2.1)
$$\begin{array}{ccccccccc} P_1(\Delta t\, L)w_{n,1} & = & Q_{11}(\Delta t\, L)w_{n-1,1} & + & Q_{12}(\Delta t\, L)w_{n-1,2} & + \cdots + & Q_{1k}(\Delta t\, L)w_{n-1,k} \\ \vdots & & \vdots & & \vdots & & \vdots \\ P_k(\Delta t\, L)w_{n,k} & = & Q_{k1}(\Delta t\, L)w_{n-1,1} & + & Q_{k2}(\Delta t\, L)w_{n-1,2} & + \cdots + & Q_{kk}(\Delta t\, L)w_{n-1,k} \end{array}$$

Here P_r and Q_{rs} are polynomials specifying the process. The coefficients of the polynomials are assumed to be real if $\mathbb{V}$ is a vector space over $\mathbb{R}$, and complex otherwise. The vectors w_{nr} $(1 \leq r \leq k)$ can be thought of as being related to the solution $U(t)$ of (1.4) at $t \approx n\,\Delta t$.

This process can be viewed as a generalization of the so-called rational k-step method, dealt with e.g. in [27]. A concrete example is provided by general Runge-Kutta methods, in which case $k = 1$, and $w_{n,1}$ approximates $U(t)$ for $t = n\,\Delta t$.

Another example is given by the LMM formula (1.5), which can be reformulated as a process of form (2.1), with

$$(2.2)\qquad \begin{array}{lll} P_1(z) = 1 - b_0 z, & Q_{1s}(z) = a_s + b_s z & (1 \le s \le k), \\ P_r(z) = 1, & Q_{r,r-1}(z) = 1, \;\; Q_{r,s}(z) = 0 & (2 \le r \le k, \;\; s \ne r-1). \end{array}$$

In this case, we have $w_{ns} = u_{n+k-s} \approx U(t)$, with $t = (n+k-s)\Delta t$.

We shall formulate conditions under which the numerical approximations w_{nr}, generated by the general process (2.1), satisfy

$$(2.3)\qquad \max_{1\le r\le k} ||w_{N,r}|| \le \mu \max_{1\le r\le k} ||w_{0,r}|| \quad (\text{whenever } w_{nr} \in \mathbb{V} \text{ satisfy (2.1) for } 1 \le n \le N),$$

with μ independent of critical variables, such as $N \ge 1$ and $\Delta t \to 0$.

To formulate these conditions concisely, we give some definitions. We will say that a matrix satisfies the *root condition* if its characteristic polynomial satisfies the root condition (as defined in Section 1.1). By $\Phi(z)$ we will denote the $k \times k$ matrix

$$\Phi(z) = (\Phi_{rs}(z)) \quad \text{with} \quad \Phi_{rs}(z) = Q_{rs}(z)/P_r(z) \quad (\text{for } 1 \le r \le k, \; 1 \le s \le k).$$

The *stability region* S, corresponding to the general process (2.1), is defined by

$$S = \{z : \; z \in \mathbb{C}, \;\; P_r(z) \ne 0 \text{ (for } 1 \le r \le k), \;\; \text{and } \Phi(z) \text{ satisfies the root condition}\}.$$

If (2.2) holds, then this set S equals the stability region of the LMM, defined in Section 1.1.

In the following theorem, constants γ_0 will occur with

$$(2.4)\qquad \{z : \; z \in \mathbb{C} \text{ with } |z + \gamma_0| \le \gamma_0\} \subset S.$$

A value γ_0 with this property (or the supremum of such values) is sometimes called *stability radius*; cf. e.g. [18], [19], [36]. Along with (2.4), constants γ will occur with

$$(2.5)\qquad 0 < \gamma < \gamma_0.$$

This is our main theorem:

Theorem 2.1. *Let polynomials P_r, Q_{rs} and a constant $\gamma > 0$ be given. Assume that, for some γ_0, conditions* (2.4) (2.5) *are fulfilled. Then there is a constant μ which does not depend on $\mathbb{V}$, L, Δt, $N \ge 1$ or $w_{0,1}, \ldots, w_{0,k}$, such that:*

(I) *The estimate* (2.3) *holds, whenever $\Delta t\, L$ satisfies the circle condition* (1.7)*;*

(II) *The estimate* (2.3) *holds, whenever there is a $\tau > 0$ such that, at the same time, L and Δt satisfy* (1.12) *and* (1.9)*, respectively.*

The theorem will be proved in Sections 4, 5.

Remark 2.2. We emphasize that the constant μ, given by Theorem 2.1, does **not** depend directly on $\mathbb{V}$, L, Δt, etc.

In fact, when initial value problem (1.4) results from semidiscretizing a partial differential equation, the operator L depends on a spacial discretization parameter, say $\Delta x > 0$, so that actually a family of operators $L = L_{\Delta x}$ is involved. In such cases it could be disastrous if μ would depend directly on L (and thus on Δx, e.g. according to $\mu \sim 1/\Delta x$ with $\Delta x \to 0$).

In case of semidiscretizations where Δt and Δx are related to each other such that (1.7) holds with $L = L_{\Delta x}$ and a fixed γ, the corresponding value μ given by Theorem 2.1 is **in**dependent of the actual values Δt, Δx.

One may wonder whether Theorem 2.1 can be improved by replacing (2.5) with

$$0 < \gamma \leq \gamma_0. \tag{2.6}$$

Such a replacement is *not* possible; the theorem is *best possible* in the following sense:

Remark 2.3. If condition (2.5) in Theorem 2.1 would be replaced by (2.6), then the theorem would no longer be true.

This remark follows from a counterexample in [19], p. 75, which shows that, under the assumptions $\gamma = \gamma_0$ and (2.4), the estimate (2.3) is *not* always present (with μ *in*dependent of $\mathbb{V}$, L, Δt, $N \geq 1$ and $w_{0,1}, \dots, w_{0,k}$) when (1.7) holds.

For completeness, we note that Theorem 2.1 could be viewed as an extension of [19] (theorem 6.2), where the case $k = 1$, $\mathbb{V} = \mathbb{R}^s$ (with maximum norm $||\cdot||_\infty$) is considered.

3. Applications to linear multistep methods

Below, we consider LMMs and denote by S the stability region as defined in Section 1.1. We make the usual assumption that, in addition to (1.3), the origin 0 belongs to S, i.e., the polynomial $\zeta^k - \sum_{j=0}^{k-1} a_{k-j}\zeta^j$ satisfies the root condition (cf. Section 1.1).

Applying Theorem 2.1, via (2.2), to k-step LMMs, we immediately obtain:

Theorem 3.1. *Let a LMM and $\gamma > 0$ be given. Assume there is a γ_0 with* (2.4)*,* (2.5)*. Then μ exists (independent of $\mathbb{V}$, L, Δt, $N \geq 1$ and $w_{0,1}, \dots, w_{0,k}$) such that:*

(I) *Estimate* (1.6) *holds, with $\alpha = 0$, whenever $\Delta t\, L$ satisfies the circle condition* (1.7)*.*

(II) *When applying the LMM to initial value problem* (1.4)*, boundedness is present in the sense of* (1.11)*, whenever* (1.12) *holds and $0 < \Delta t \leq \gamma\,\tau$.*

With an eye to the role played by γ in Statement (II) of this theorem, a value $\gamma > 0$ will be called a *stepsize-coefficient for linear boundedness* of a LMM, if a constant μ exists (independent of $\mathbb{V}$, L, Δt, $N \geq 1$ and $w_{0,1}, \dots, w_{0,k}$) such that (1.11) holds whenever the LMM is applied to any problem (1.4) under conditions (1.12), (1.9). Clearly, by Theorem 3.1, conditions (2.4), (2.5) imply that γ is such a stepsize-coefficient.

In the present context, the so-called *growth parameters* of the LMM (cf. e.g. [11]) are useful. To specify them, we put $\rho(\zeta) = \zeta^k - \sum_{j=0}^{k-1} a_{k-j}\zeta^j$, $\sigma(\zeta) = \sum_{j=0}^{k} b_{k-j}\zeta^j$, and denote the roots of $\rho(\zeta)$ with modulus equal to 1, by $\eta_1, \dots, \eta_q$. We choose the numbering such that $\eta_1 = 1$, which is possible by (1.3). The *growth parameters* $\lambda_1, \dots, \lambda_q$, are defined by

$$\lambda_j = \frac{\sigma(\eta_j)}{\eta_j \cdot \rho'(\eta_j)}, \tag{3.1}$$

so that $\lambda_1 = 1$, by (1.3). By expanding the roots $\zeta \approx \eta_j$ of $P(\zeta) = \rho(\zeta) - z\,\sigma(\zeta)$ (for $z \approx 0$) in powers of z (cf. e.g. [3] (chapter 1)), the following lemma can be proved.

TABLE 1. Values of k, for which stepsize-coefficients γ exist relevant to: linear boundedness (1.4), (1.12), (1.11), linear monotonicity (1.4), (1.12), (1.10), general boundedness (1.1), (1.8), (1.11) and general monotonicity (1.1), (1.8), (1.10).

		A-M	*A-B*	*BD*	*EBD*	*M-S*	*N*
1.	*Linear Boundedness*	all $k \geq 1$	all $k \geq 1$	$1 \leq k \leq 6$	$1 \leq k \leq 6$	none	none
2.	*Linear Monotonicity*	$k = 1$	$k = 1$	$k = 1$	$k = 1$	none	none
3.	*General Boundedness*	$1 \leq k \leq 8$	$1 \leq k \leq 3$	$1 \leq k \leq 6$	$1 \leq k \leq 5$	none	none
4.	*General Monotonicity*	$k = 1$	$k = 1$	$k = 1$	$k = 1$	none	none

Lemma 3.2. *There exists a value* $\gamma_0 > 0$ *with property* (2.4)*, if and only if all growth parameters* λ_j *are real and non-negative.*[3]

Combining Theorem 3.1 and this lemma, we arrive at:

Corollary 3.3.

(I) *For any LMM, there exists a stepsize-coefficient* γ *for linear boundedness, if and only if all growth parameters* λ_j *of the method are real and non-negative.*

(II) *If* $\zeta = 1$ *is the only root with modulus one of the polynomial* $\rho(\zeta)$*, then there exists a stepsize-coefficient for linear boundedness.*

Proof. (I) If all λ_j are real and non-negative, the conclusion follows from Lemma 3.2 and Theorem 3.1 (e.g. with $\gamma = \gamma_0/2$). Conversely, if γ is a stepsize-coefficient for linear boundedness, then boundedness must be present for the special case where $\mathbb{V} = \mathbb{C}$ and $\Delta t\, L = -\gamma + \gamma \cdot \theta$, with $\theta \in \mathbb{C}$ and $|\theta| \leq 1$. This means $1 - b_0\,(-\gamma + \gamma\,\theta) \neq 0$ and the polynomial $\rho(\zeta) - (-\gamma + \gamma \cdot \theta)\,\sigma(\zeta)$ satisfies the root condition. Hence $\{z :\ z \in \mathbb{C} \text{ with } |z + \gamma| \leq \gamma\} \subset S$. Applying Lemma 3.2 with $\gamma_0 = \gamma > 0$, it follows that all λ_j are real and non-negative.

(II) Part (II) follows from Part (I), because $\lambda_1 = 1 > 0$. □

Part (II) of this corollary is relevant to all *Adams-Bashforth* (A-B) and *Adams-Moulton* (A-M) methods; as well as (for $1 \leq k \leq 6$) to *backward differentiation* (BD) methods and *extrapolated versions* (EBD) thereof: for all of these methods, a stepsize-coefficient exists for linear boundedness. On the other hand, for the *Milne-Simpson* (M-S) and *Nyström* (N) methods, Part (I) of Corollary 3.3 can be applied with $k \geq 2$ and $\lambda_2 < 0$: within neither of these classes, a stepsize-coefficient for linear boundedness exists. All of these conclusions are given in Line 1 (indicated with *Linear Boundedness*) of Table 1.[4]

Definitions analogous to the above definition of a *stepsize-coefficient for linear boundedness*, can be given with regard to case (1.4, 1.12, 1.10), case (1.1, 1.8, 1.11) and case (1.1, 1.8, 1.10), so that in total four kinds of stepsize-coefficients are worth considering. We have included results for all of these stepsize-coefficients in the table, allowing a neat comparison with earlier results, in the literature, about monotonicity/boundedness. The entries in Lines 2 and 4 (*Linear Monotonicity* and *General Monotonicity*) refer to monotonicity property (1.10), in case of (1.4) and (1.1), respectively; the indicated ranges of k follow e.g. from [20], [35] (p. 283). The

[3]This condition on the growth parameters reduces to positivity of all λ_j, under the assumption that $\rho(\zeta)$ and $\sigma(\zeta)$ have no common root. A corresponding version of Lemma 3.2 was given in [3] (chapter 1, theorem 4.5.)

[4]For definitions and details of the six classes of LMMs considered, see e.g. [8], [11], [17], [39].

entries in Line 3 (*General Boundedness*) refer to boundedness property (1.11) in solving the general problem (1.1) – the given ranges for k are taken from [39].

4. Preliminaries to the proof of Theorem 2.1

In this section, we shall make two observations simplifying the actual proof of Theorem 2.1.

Our first observation concerns the connection between Statements (I) and (II) of Theorem 2.1. In fact, only one of these statements has to be proved. This follows from the close relation of condition (1.7) to conditions (1.12), (1.9). This relation was observed by various authors, and is formulated explicitly below.

Lemma 4.1. *Let L, $\Delta t > 0$ and $\gamma > 0$ be given. Then the circle condition* (1.7) *is in force, if and only if a value $\tau > 0$ exists, for which both* (1.12) *and* (1.9) *are fulfilled.*

Proof of Lemma 4.1. To prove the lemma, note first that (1.12), (1.9) imply:

$$||(\Delta t\, L + \gamma I)\, v_0|| = \gamma \cdot ||(I + \frac{\Delta t}{\gamma} L)\, v_0|| \leq \gamma ||v_0||,$$

i.e., (1.7).

Next, assuming (1.7), we define $\tau = \Delta t/\gamma$, so that (1.9) holds. If $0 < \tau_0 \leq \tau$, then: $||v_0 + \tau_0 L v_0|| = ||\frac{\tau_0}{\Delta t}(\Delta t\, L + \gamma) v_0 + (1 - \frac{\gamma\, \tau_0}{\Delta t}) v_0|| \leq \frac{\tau_0}{\Delta t}\gamma ||v_0|| + (1 - \frac{\gamma\, \tau_0}{\Delta t})||v_0|| = ||v_0||$, i.e., (1.12) □

Our second observation concerns the framework used thus far, viz:

(4.1) $\quad\mathbb{V}$ is an arbitrary vectorspace with seminorm $|| \cdot ||$.

In fact, the following lemma shows that, for proving the theorem in that general framework, it is enough to prove it in the following more restricted setting:

(4.2) $\quad\mathbb{V}$ is a complex Banach space with norm $|| \cdot ||$.

Lemma 4.2. *Let polynomials P_r, Q_{rs} be given. Assume that, for some constants γ, μ, the implication "*(1.7) $\Longrightarrow$ (2.3)*" is valid in the setting* (4.2).

Then, with the same constants, the implication is valid in the setting (4.1) *as well.*

Proof of Lemma 4.2. We shall first prove the implication "(1.7) $\Longrightarrow$ (2.3)", in the following setting:

(4.3) $\quad\mathbb{V}$ is a complex vectorspace with seminorm $|| \cdot ||$.

In the situation (4.3), we can construct a space $\overline{\mathbb{V}}$ and function f such that

(4.4) $\quad\overline{\mathbb{V}}$ is a complex Banach space with norm $|| \cdot ||$,

(4.5) $\quad f$ is a linear function from V into $\overline{\mathbb{V}}$, with $||f(v)|| = ||v||$ (for all $v \in \mathbb{V}$),

(4.6) $\quad f[\mathbb{V}]$ is dense in $\overline{\mathbb{V}}$.

The space $\overline{\mathbb{V}}$ and function f can be obtained, e.g., by defining $f(v) = \phi(\psi(v))$ (for $v \in \mathbb{V}$). Here ψ denotes the quotient mapping from $\mathbb{V}$ onto the quotient space $\mathbb{V}^* = \mathbb{V}/\mathbb{V}_0$, where $\mathbb{V}_0 = \{v : v \in \mathbb{V} \text{ with } ||v|| = 0\}$ and $\psi(v) = v + \mathbb{V}_0$, $||\psi(v)|| = ||v||$. Furthermore, ϕ is a linear function from the normed vectorspace $\mathbb{V}^*$ into a Banach space $\overline{\mathbb{V}}$, such that $||\phi(w)|| = ||w||$ (for all $w \in \mathbb{V}^*$), and $\phi[\mathbb{V}^*]$

is dense in $\overline{\mathbb{V}}$. Here $\overline{\mathbb{V}}$ denotes a standard completion (of the normed linear space $\mathbb{V}^*$); cf. e.g. [40], sections 3.13, 3.14.

Assume (4.3), and let $L : \mathbb{V} \to \mathbb{V}$ be a linear operator satisfying (1.7). Let $w_{n,r} \in \mathbb{V}$ satisfy (2.1) (for $1 \leq n \leq N$).

We define $\overline{L} : f[\mathbb{V}] \to f[\mathbb{V}]$ by $\overline{L}(f(v)) = f(L\,v)$; which can be seen to be an unambiguous definition by using (4.5) and the fact that L is a linear operator satisfying $||Lv|| \leq \beta ||v||$ (for all $v \in \mathbb{V}$) with some fixed β.

Using (4.6), we extend $\overline{L}$ to a linear operator from $\overline{\mathbb{V}}$ to itself satisfying (1.7) (where $\overline{L}$, $\overline{\mathbb{V}}$ now replace L, $\mathbb{V}$). For $1 \leq n \leq N$, the vectors $\overline{w}_{nr} = f(w_{nr}) \in \overline{\mathbb{V}}$ can be seen to satisfy (2.1) (where $\overline{L}$, $\overline{w}_{nr}$ now replace L, w_{nr}).

Because the implication "(1.7)$\Longrightarrow$(2.3)" holds in the complex Banach space $\overline{\mathbb{V}}$, we have

$$\max_{1 \leq r \leq k} ||w_{N,r}|| = \max_{1 \leq r \leq k} ||\overline{w}_{N,r}|| \leq \mu \max_{1 \leq r \leq k} ||\overline{w}_{0,r}|| = \mu \max_{1 \leq r \leq k} ||w_{0,r}||,$$

which proves that the implication holds in the framework (4.3) as well.

It remains to show that the implication "(1.7)$\Longrightarrow$(2.3)" is valid, also in the case where

(4.7) $\mathbb{V}$ is a real vectorspace with seminorm $||\cdot||$.

In the situation (4.7), we consider the complexification $\widehat{\mathbb{V}} = \mathbb{V} \oplus \mathbb{V}$, equipped with the seminorm $||w|| = \sup_{t \in \mathbb{R}} ||\cos(t)\,u + \sin(t)\,v||$ for $w = (u, v) \in \widehat{\mathbb{V}}$; cf. [25].

Assuming (4.7), and L, w_{nr} to be as in (1.7), (2.1) (for $1 \leq n \leq N$), we define $\widehat{L}$ by the rule $\widehat{L}(u, v) = (Lu, Lv)$ (for $(u, v) \in \widehat{V}$), and we write $\widehat{w}_{nr} = (w_{nr}, 0)$. Using the implication "(1.7)$\Longrightarrow$(2.3)" in the space $\widehat{\mathbb{V}}$, the implication can be seen to hold in the space $\mathbb{V}$ as well. □

The above two lemmas show that for establishing Theorem 2.1, it is enough to prove just Statement (I) of Theorem 2.1 in the context of complex Banach spaces.

5. Proving Theorem 2.1, (I), for complex Banach spaces

5.1. **Part 1 of the proof.** Throughout Section 5, we denote by $\mathbb{V}$ a complex Banach space with norm $||\cdot||$, and we make, unless stated otherwise, the assumptions (2.4), (2.5).

For $\zeta \in \mathbb{C}^k$, with components $\zeta_1, \ldots, \zeta_k$, we use the norm $|\zeta| = \max\,\{|\zeta_r| \,:\, 1 \leq r \leq k\}$. For $k \times k$ matrices A, we put $|A| = \max\{|A\zeta|/|\zeta| \,:\, \zeta \in \mathbb{C}^k,\, \zeta \neq 0\}$. For $w \in \mathbb{V}^k$, with components $w_1, \ldots, w_k \in \mathbb{V}$, we use the norm

$$||w|| = \max\,\{||w_r|| \,:\, 1 \leq r \leq k\}.$$

Let A be a $k \times k$ matrix. For linear operators $X : \mathbb{V} \to \mathbb{V}$, we denote by $A \otimes X$ the operator mapping $u \in \mathbb{V}^k$ with components u_ℓ (for $1 \leq \ell \leq k$), into $v \in \mathbb{V}^k$ with components $v_i = \sum_{\ell=1}^{k} a_{i\ell} X u_\ell$ (for $1 \leq i \leq k$). Below it will be used that

(5.1) $$||(A \otimes Y^j)\,w|| \leq |A|\,||w|| \quad \text{(for all } w \in \mathbb{V}^k \text{ and } j = 0, 1, 2, \ldots), \\ \text{if} \quad ||Y\,v|| \leq ||v|| \;\; \text{(for all } v \in \mathbb{V}).$$

Because of (2.5), (2.4), there is a value $\sigma > 0$ such that

(5.2) $$P_r(z) \neq 0 \quad \text{for } 1 \leq r \leq k \text{ and all } z \in \mathbb{C} \text{ with } |z + \gamma| \leq (1 + \sigma)\gamma.$$

The functions $\Phi_{rs}(z) = \frac{Q_{rs}(z)}{P_r(z)}$ are thus holomorphic for $z \in \mathbb{C}$ with $|z+\gamma| \le (1+\sigma)\gamma$, and

$$\Phi(-\gamma + \gamma y)^n = \sum_{j=0}^{\infty} y^j\, C_{nj} \ \ (\text{for } |y| \le 1+\sigma), \quad \text{with } C_{nj} = \frac{\gamma^j}{j!} \left\{ \frac{\mathrm{d}^j}{\mathrm{d}z^j} \big[\Phi(z)^n\big] \right\}_{z=-\gamma}.$$

Assume the circle condition (1.7) is fulfilled. In dealing with (2.1), we shall use the following operator Z and (column) vectors $w_n \in \mathbb{V}^k$:

(5.3) $$Z = \Delta t\, L \quad \text{and} \quad w_n \in \mathbb{V}^k \text{ with components } w_{n1}, \dots w_{nk}.$$

Clearly,

$$||(Z + \gamma I)\, v|| \le \gamma\, ||v|| \text{ (for all } v \in \mathbb{V}).$$

The last inequality implies that $Z = -\gamma\, I + \gamma\, Y$, with $||Y\, v|| \le ||v||$ (for all $v \in \mathbb{V}$).

Because $P_r(z) \ne 0$ on the disk $\{z : |z+\gamma| \le \gamma\}$, the roots of $P_r(z)$ belong to the resolvent set of Z. Consequently, the operators $P_r(Z)$ have inverses $P_r(Z)^{-1}$ mapping $\mathbb{V}$ into itself. For given $w_0 \in \mathbb{V}^k$, there are thus unique w_n satisfying (2.1) (for $1 \le n \le N$); clearly, $w_{nr} = P_r(Z)^{-1}Q_{r1}(\Delta t\, L)w_{n-1,1} + P_r(Z)^{-1}Q_{r2}(\Delta t\, L)w_{n-1,2} + \cdots + P_r(Z)^{-1}Q_{rk}(\Delta t\, L)w_{n-1,k}$.

We have $P_r(Z)^{-1}Q_{rs}(Z) = \Phi_{rs}(Z)$, where $\Phi_{rs}(Z)$ is defined via the partial fraction decomposition of $\Phi_{rs}(z)$, or (equivalently) via a suitable contour integral of the operator-valued function $f(\lambda) = \Phi_{rs}(\lambda)\,(\lambda I - Z)^{-1}$; see e.g. [40] (section 5.6) or [5] (p. 11).

Assume (2.1) (for $1 \le n \le N$). Defining the operator $\Phi(Z) = (\Phi_{rs}(Z))$ (from $\mathbb{V}^k$ to itself) in a natural way, we get $w_N = \Phi(Z)^N w_0 = \Phi(-\gamma I + \gamma\, Y)^N w_0 = \left\{\sum_{j=0}^{\infty} C_{Nj} \otimes Y^j\right\} w_0$; see e.g. [5] (theorem 1.19). Applying (5.1), we arrive in view of the above at:

Lemma 5.1. *Assume* (1.7)*, and* $w_0 \in \mathbb{V}^k$*,* $N \ge 1$*. Then there are unique* w_n *satisfying* (2.1) *(for* $1 \le n \le N$*). Moreover,*

(5.4) $$||w_N|| \le \Big(\sum_{j=0}^{\infty} |C_{Nj}|\Big)\, ||w_0||.$$

For proving Theorem 2.1, (I), via (5.4), we have to find μ (not depending on n) with

(5.5) $$\sum_{j=0}^{\infty} |C_{nj}| \le \mu \quad (\text{for } n \ge 1).$$

Clearly, $C_{nj} = \frac{1}{2\pi \mathrm{i}} \oint_{|y|=1} y^{-j-1}\, \Phi(-\gamma + \gamma y)^n \mathrm{d}y = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}jt}\, \Phi(-\gamma + \gamma\, \mathrm{e}^{-\mathrm{i}t})^n \mathrm{d}t$. Defining

$$F(t) = \Phi(-\gamma + \gamma\, \mathrm{e}^{-\mathrm{i}t}),$$

we thus have

$$C_{nj} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}jt}\, F(t)^n \mathrm{d}t.$$

Below, in Section 5.2, we shall split the powers $F(t)^n$ into a sum

$$F(t)^n = F_n^{[0]}(t) + F_n^{[1]}(t) + \dots F_n^{[s]}(t), \tag{5.6}$$

where each function $F_n^{[\ell]}(t)$ has a more simple structure than $F(t)^n$.

Next, in Sections 5.3 and 5.4, we shall prove for the matrices

$$C_{nj}^{[\ell]} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}jt}\, F_n^{[\ell]}(t)\mathrm{d}t, \tag{5.7}$$

that finite constants M_ℓ, not depending on n, exist with

$$\sum_{j=0}^{\infty} |C_{nj}^{[\ell]}| \leq M_\ell \quad (\text{for } n \geq 1). \tag{5.8}$$

These bounds will complete the proof, because they imply that $\sum_{j=0}^{\infty} |C_{nj}| \leq \sum_{j=0}^{\infty}(|C_{nj}^{[0]}| + \dots + |C_{nj}^{[s]}|) \leq \mu$ (for $n \geq 1$), with $\mu = M_0 + M_1 + \dots + M_s$.

5.2. **Part 2 of the proof: defining the functions $F_n^{[0]}(t), \dots, F_n^{[s]}(t)$.** We will denote the *spectrum* of any matrix A in $\mathbb{C}^{k\times k}$ by sp$[A]$, and its *spectral radius* by spr$[A]$. Furthermore, an eigenvalue of A will be said to have multiplicity m, if it is a root of the characteristic polynomial with multiplicity m.

The following lemma will be used repeatedly:

Lemma 5.2. *Assume $|z^* + \gamma_0| \leq \gamma_0$, and λ^* is an eigenvalue of $\Phi(z^*)$ with multiplicity m. Let $\delta > 0$, $\epsilon > 0$. Then the following holds:*

(I) *There are $\delta^* \in (0, \delta)$, $\epsilon^* \in (0, \epsilon)$, such that for each z with $|z - z^*| < \epsilon^*$, there are precisely m eigenvalues λ of $\Phi(z)$ with $|\lambda - \lambda^*| < \delta^*$; each eigenvalue being counted according to its multiplicity.*

Assume, in addition to the above, that $|\lambda^*| = 1$. Then Statements (IIa)–(IId) *hold:*

(IIa) *There are $\delta^* \in (0, \delta)$, $\epsilon^* \in (0, \epsilon)$ and α_j such that, for each z with $|z - z^*| < \epsilon^*$,*

$\rhd$ *the power series $1 + \alpha_1(z - z^*) + \alpha_2(z - z^*)^2 + \dots$ converges;*

$\rhd$ *there is precisely one (simple) eigenvalue λ of $\Phi(z)$ with $|\lambda - \lambda^*| < \delta^*$;*

$\rhd$ *the last mentioned eigenvalue equals $\lambda = \lambda^*[1 + \alpha_1(z - z^*) + \alpha_2(z - z^*)^2 + \dots]$.*

(IIb) *If, in statement* (IIa), *all $\alpha_j = 0$, then $\lambda^* \in \mathrm{sp}[\Phi(z)]$ for all z with $|z + \gamma_0| \leq \gamma_0$.*

(IIc) *If $|z^* + \gamma| \leq \gamma$, $z^* \neq 0$, then, in statement* (IIa), *we have all $\alpha_j = 0$.*

(IId) *Let $z^* = 0$. If statement* (IIa) *holds with not all $\alpha_j = 0$, then α_1 is real and positive.*

Proof. (I) Define $P(z, \lambda) = \det[\Phi(z) - \lambda I]$ and $f(\lambda) = P(z^*, \lambda)$. For $\delta^* \in (0, \delta)$ small enough, the disk $\{\lambda : |\lambda - \lambda^*| \leq \delta^*\}$ contains no zeros of $f(\lambda)$ other than the (m-fold) zero $\lambda = \lambda^*$. Therefore, Statement (I), follows e.g. by applying Rouché's theorem, with function $g(\lambda) = P(z, \lambda)$ satisfying

$$|g(\lambda) - f(\lambda)| < |f(\lambda)| \quad \text{for } |\lambda - \lambda^*| = \delta^* \text{ and } |z - z^*| \leq \epsilon^*,$$

where $\epsilon^* \in (0, \epsilon)$ is sufficiently small; cf. e.g [31], p. 242.

(IIa) The assertion in Statement (IIa), *not* dealing with the power series, follows from Statement (I) (with $m = 1$). Furthermore, with $P(z, \lambda)$ as defined above, we have $P(z^*, \lambda^*) = 0$, $\frac{\partial}{\partial\lambda} P(z^*, \lambda^*) \neq 0$. Therefore, the assertions about the power series follow from the expansion theorem as given e.g. in [1], p. 17.

(IIb) Because $P(z, \lambda^*) = 0$ for all z in a neighbourhood of z^*, we must have $P(z, \lambda^*) = 0$ for all z with $|z + \gamma_0| \leq \gamma_0$.

(IIc) Suppose there would be an $\alpha_j \neq 0$. We can choose an open neighbourhood of z^* lying in the stability region S. The intersection of that neighbourhood and the open disk $\{z : \; |z - z^*| < \epsilon^*\}$ (where ϵ^* is as in Statement (IIa)), is mapped by the function $\phi(z) = \lambda^*[1 + \alpha_1(z - z^*) + \alpha_2(z - z^*)^2 + \dots]$ onto an open neighbourhood of λ^*. This would imply that there are points z in S with $\mathrm{spr}[\Phi(z)] > 1$; a contradiction.

(IId) *Part 1.* Let $z^* = 0$, and $\delta^*, \epsilon^*, \alpha_j$ as in (IIa), with not all $\alpha_j = 0$. We claim that $\alpha_1 \neq 0$. Suppose, to the contrary, that α_m is the first coefficient with $\alpha_m \neq 0$, and $m \geq 2$.

Let $0 < \eta < \frac{1}{2}$. For a radius $r > 0$, to be specified below, we consider the curve

$$\Gamma : \; t \longrightarrow z = r \, \exp(\mathrm{i}\, t), \qquad \text{for } (\tfrac{1}{2} + \eta)\pi \leq t \leq (\tfrac{3}{2} - \eta)\pi.$$

We denote the corresponding range by $|\Gamma| = \{z : \; z = r \, \exp(\mathrm{i}\, t), \; (\frac{1}{2} + \eta)\pi \leq t \leq (\frac{3}{2} - \eta)\pi\}$.

There is an $r_0 > 0$ such that, for all $r \in (0, r_0]$,

$$|\Gamma| \text{ is contained in the stability region } S.$$

The function $\phi(z) = \lambda^*[1 + \alpha_1 \, z + \alpha_2 \, z^2 + \dots]$ satisfies

$$\phi(z) = \lambda^* \left[1 + \alpha_m(1 + \mathcal{O}(z))z^m\right] \quad (\text{for } z \to 0).$$

Hence, there is an $r_1 \in (0, \epsilon^*)$ such that for all $r \in (0, r_1]$ and $|z| = r$ we have

$$\phi(z) = \lambda^* \left[1 + \alpha_m(1 + \psi(z))z^m\right] \quad \text{with } |\psi(z)| < 1, \; \left|\arg[1 + \psi(z)]\right| \leq \eta \, \pi.$$

Denoting the increase of the argument of any function $\chi(z)$, when z runs through the curve Γ, by $[\arg\{\chi(z)\}]_\Gamma$, we thus have, for $0 < r \leq r_1$:

$$\begin{aligned} \left[\arg\left\{\frac{\phi(z)}{\lambda^*} - 1\right\}\right]_\Gamma &= [\arg\{1 + \psi(z)\}]_\Gamma + [\arg\{z^m\}]_\Gamma \\ &\geq -2\,\eta\pi + m\,(1 - 2\,\eta)\,\pi \geq (2 - 6\,\eta)\pi. \end{aligned}$$

We choose η with $0 < \eta < \frac{1}{6}$, and consider related values r_0, r_1. We put $r = \min\{r_0, r_1\}$, so that for the corresponding Γ we have $\left[\arg\left\{\frac{\phi(z)}{\lambda^*} - 1\right\}\right]_\Gamma > \pi$. There is thus a point $z_0 \in |\Gamma|$ with $\mathrm{Re}\left\{\frac{\phi(z_0)}{\lambda^*} - 1\right\} > 0$, and $\left|\frac{\phi(z_0)}{\lambda^*}\right| \geq \left|\mathrm{Re}\left\{\frac{\phi(z_0)}{\lambda^*}\right\}\right| = \left|1 + \mathrm{Re}\left\{\frac{\phi(z_0)}{\lambda^*} - 1\right\}\right| > 1$. As $z_0 \in |\Gamma| \subset S$ and $\phi(z_0) \in \mathrm{sp}[\Phi(z_0)]$, we have $\left|\frac{\phi(z_0)}{\lambda^*}\right| \leq 1$, yielding a contradiction. Hence,

$$\alpha_1 \neq 0.$$

(IId) *Part 2.* To prove $\alpha_1 > 0$, we consider $z \to 0$ with $|z + \gamma| \leq \gamma$. Because $z \in S$, we have, with the same notation as above:

$$1 \geq \left|\frac{\phi(z)}{\lambda^*}\right| \geq 1 + \mathrm{Re}\left\{\frac{\phi(z)}{\lambda^*} - 1\right\} = 1 + \mathrm{Re}\big\{\alpha_1 z(1 + \mathcal{O}(z))\big\} \quad (\text{for } z \to 0).$$

Therefore,

$$\mathrm{Re}\{\alpha_1 z\} \leq \mathcal{O}(|z|^2) \quad (\text{when } z \to 0, \text{ while } |z + \gamma| \leq \gamma).$$

Writing $\alpha_1 = |\alpha_1|\mathrm{e}^{\mathrm{i}\theta_1}$ and $z = |z|\mathrm{e}^{\mathrm{i}\theta(z)}$, it follows that:

$$|\alpha_1| \cos(\theta_1 + \theta(z)) \leq \mathcal{O}(|z|) \quad (\text{when } z \to 0, \text{ while } |z + \gamma| \leq \gamma). \tag{5.9}$$

We let z tend to zero (while $|z+\gamma| \le \gamma$) in three different manners, viz. such that $\theta(z) \equiv \pi$, and such that $\theta(z) \to \pi/2$, as well as $\theta(z) \to -\pi/2$. This leads, respectively, to

$$\cos(\theta_1) \ge 0, \quad \sin(\theta_1) \ge 0, \quad \sin(\theta_1) \le 0.$$

Hence, $\cos(\theta_1) = 1$, i.e., $\alpha_1 > 0$. □

Our definition of the function F implies that $F(t) = \Phi(z)$ with $|z+\gamma| \le \gamma$. Hence, in view of Parts (IIc), (IId) of Lemma 5.2, the eigenvalues λ^* of $F(t)$ with modulus $|\lambda^*| = 1$ fit into two separate categories.

The first category consists of the eigenvalues (of unit modulus) of $F(0)$ for which the coefficient $\alpha_1 > 0$. We denote these eigenvalues by

$$\lambda_1^*, \dots, \lambda_p^*.$$

The second category consists of the eigenvalues (of unit modulus) of $F(t)$ for which all coefficients $\alpha_j = 0$. We denote them by

$$\lambda_{p+1}^*, \dots, \lambda_{p+q}^*.$$

We choose $\delta_0 > 0$ so small that

$$\text{all disks } \{\lambda : |\lambda - \lambda_\ell^*| \le \delta_0\} \text{ are disjoint } (1 \le \ell \le p+q). \tag{5.10}$$

From Lemma 5.2 (and a compactness argument), one arrives at

Remark 5.3.

(I) There is a δ_1 with $0 < \delta_1 \le \delta_0$, such that for $0 \le |t| \le \pi$, and $p+1 \le \ell \le p+q$, the only $\lambda \in \mathrm{sp}[F(t)]$ with $|\lambda - \lambda_\ell^*| \le \delta_1$, equals λ_ℓ^*.

(II) For $0 < |t| \le \pi$, all eigenvalues λ of $F(t)$ that are different from $\lambda_{p+1}^*, \dots, \lambda_{p+q}^*$, have modulus $|\lambda| < 1$.

(III) All eigenvalues λ of $F(0)$ that are different from $\lambda_1^*, \dots, \lambda_p^*, \lambda_{p+1}^*, \dots, \lambda_{p+q}^*$, have a modulus $|\lambda| < 1$.

We can conclude, from the Jordan canonical form of $F(t)$, that, for $0 \le |t| \le \pi$:

$$F(t) = P(t) + Q(t), \text{ with } P(t)Q(t) = Q(t)P(t) = 0,$$
$$\mathrm{sp}[P(t)] = \mathrm{sp}[F(t)] \setminus \{\lambda_{p+1}^*, \dots, \lambda_{p+q}^*\}, \quad \mathrm{sp}[Q(t)] = \{\lambda_{p+1}^*, \dots, \lambda_{p+q}^*\},$$
$$Q(t) = F_{p+1}(t) + \dots + F_{p+q}(t), \text{ with } F_\ell(t)F_m(t) = 0 \text{ (for } \ell \ne m),$$
$$F_\ell(t)^n = (\lambda_\ell^*)^{n-1} F_\ell(t) \text{ (for } n \ge 1 \text{ and } p+1 \le \ell \le p+q).$$

A decomposition of $P(t)$, analogous to the one just given for $Q(t)$, can be obtained for $t \approx 0$, using Lemma 5.2, with $z = -\gamma + \gamma \mathrm{e}^{-it} \approx 0$. There are, for $t \to 0$, exactly p simple eigenvalues $\lambda_1(t), \dots, \lambda_p(t)$ of $P(t)$ tending to $\lambda_1^*, \dots, \lambda_p^*$, respectively. The other eigenvalues of $P(t)$ have a modulus bounded away from one. It follows that there are ε, δ with

$$0 < \delta < \min\{1, \delta_1\}, \quad 0 < \varepsilon < \pi/2,$$

and holomorphic functions $\lambda_\ell(t)$ $(1 \le \ell \le p)$, such that for $|t| \le 2\varepsilon$:

$$\mathrm{sp}[P(t)] = \Lambda_0(t) \cup \Lambda_1(t), \quad \text{where all } \lambda \in \Lambda_0(t) \text{ have a modulus } |\lambda| < 1 - \delta;$$
$$\Lambda_1(t) = \{\lambda_1(t), \dots, \lambda_p(t)\}, \quad \text{where for } 1 \le \ell \le p:$$
$$|\lambda_\ell(t) - \lambda_\ell^*| < \delta, \ |\lambda_\ell(t)| < 1 \text{ (for } t \ne 0) \text{ and } \lambda_\ell(t) \to \lambda_\ell^* \text{ (for } t \to 0).$$

Here δ_1 is as specified in Remark 5.3, and the inequality $|\lambda_\ell(t)| < 1$ is a consequence of the second statement in that remark.

It follows, from the Jordan canonical form of $P(t)$, that we can write, for $|t| \le 2\varepsilon$:

$$P(t) = F_0(t) + F_1(t) + \cdots + F_p(t), \quad \text{with} \quad F_\ell(t)F_m(t) = 0 \quad (\text{for } \ell \ne m),$$
$$\operatorname{spr}[F_0(t)] < 1-\delta, \quad F_\ell(t)^n = \lambda_\ell(t)^{n-1}F_\ell(t) \quad (\text{for } n \ge 1 \text{ and } 1 \le \ell \le p).$$

In our splitting (5.6), we shall make use of ε, δ with the properties just mentioned, and the integer s will be equal to $s = p+q$. The function $F_n^{[0]}(t)$ will be related to eigenvalues of $F(t)$ having a modulus bounded away from one. For $1 \le \ell \le p$ and $t \to 0$, the $F_n^{[\ell]}(t)$ will be related to $\lambda_\ell(t)$; and the remaining functions $F_n^{[\ell]}(t)$ to λ_ℓ^*.

We shall use a partition of unity on $[-\pi, \pi]$, involving a real-valued and twice continuously differentiable function $\phi(t)$, satisfying

$$\phi(t) = 0 \quad (2\varepsilon \le |t| \le \pi), \quad 0 \le \phi(t) \le 1 \quad (\varepsilon \le |t| \le 2\varepsilon), \quad \phi(t) = 1 \quad (|t| \le \varepsilon).$$

For formal reasons, only, we define for $2\varepsilon < |t| \le \pi$:

$$F_0(t) = \cdots = F_p(t) = 0, \quad \lambda_1(t) = \cdots = \lambda_p(t) = 0.$$

For $0 \le |t| \le \pi$, we have the decompositions

$$\begin{aligned}
F(t)^n &= \big(1-\phi(t)^n\big)P(t)^n + \phi(t)^n P(t)^n + Q(t)^n,\\
\phi(t)^n P(t)^n &= [\phi(t)F_0(t)]^n + [\phi(t)F_1(t)]^n + \cdots + [\phi(t)F_p(t)]^n,\\
Q(t)^n &= F_{p+1}(t)^n + \cdots + F_{p+q}(t)^n,
\end{aligned}$$

so that the splitting (5.6) is in force with

$$F_n^{[0]}(t) = \big(1-\phi(t)^n\big)P(t)^n + [\phi(t)\, F_0(t)]^n, \tag{5.11}$$

$$F_n^{[\ell]}(t) = [\phi(t)\, F_\ell(t)]^n \quad (1 \le \ell \le p), \quad F_n^{[\ell]}(t) = F_\ell(t)^n \quad (p+1 \le \ell \le s = p+q). \tag{5.12}$$

Because the functions $F_n^{[\ell]}(t)$ are composed of the n-th powers of matrices $P(t)$, $\phi(t)F_\ell(t)$ and $F_\ell(t)$, the subsequent remark is of importance; it will be used in the following sections.

Remark 5.4. The matrix-valued functions $P(t)$, $\phi(t)F_\ell(t)$ (for $0 \le \ell \le p$) and $F_\ell(t)$ (for $p+1 \le \ell \le p+q$) are twice continuously differentiable on $[-\pi, \pi]$. Moreover, these functions and their first derivatives assume at $t = \pi$ the same values as at $t = -\pi$.

These properties follow from $P(t) = F(t) - \big(F_{p+1}(t) + \cdots + F_{p+q}(t)\big)$ (for $0 \le |t| \le \pi$),

$$F_\ell(t) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - \lambda_\ell^*| = \delta} \zeta\, [\zeta - F(t)]^{-1} \mathrm{d}\zeta \quad (\text{for } p+1 \le \ell \le p+q \text{ and } 0 \le |t| \le \pi),$$

and the following representations, which are valid (only) for $|t| \le 2\varepsilon$:

$$\begin{aligned}
F_0(t) &= \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta| = 1-\delta} \zeta\, [\zeta - F(t)]^{-1} \mathrm{d}\zeta,\\
F_\ell(t) &= \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - \lambda_\ell^*| = \delta} \zeta\, [\zeta - F(t)]^{-1} \mathrm{d}\zeta \quad (1 \le \ell \le p).
\end{aligned}$$

5.3. **Part 3 of the proof: bounding $\sum_{j=0}^{\infty}\left|C_{nj}^{[\ell]}\right|$ $(\ell = 0,\ p+1 \le \ell \le p+q)$.** In bounding $\left|C_{nj}^{[0]}\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}jt}\, F_n^{[0]}(t)\mathrm{d}t\right|$, we shall use the following lemma, involving a function $H(t)$ defined on a finite union T of bounded closed real intervals. The function has values in the space $\mathbb{C}^{k\times k}$ of $k\times k$ matrices, and it will be assumed that

$$\mathrm{spr}[H(t)] < 1 \text{ for } t \in T, \quad \text{and} \quad H(t) \text{ has a continuous second derivative on } T.$$

Lemma 5.5. *Under the above assumptions on $H(t)$, there exist constants K and θ, with $0<\theta<1$, such that uniformly for all $n \ge 1$ and $t \in T$:*

$$\left|H(t)^n\right| \le K\,\theta^n, \quad \left|\frac{d}{dt}\{H(t)^n\}\right| \le K\,\theta^n, \quad \left|\frac{d^{\,2}}{dt^2}\{H(t)^n\}\right| \le K\,\theta^n.$$

Proof. By a compactness argument, there is an $\alpha \in (0,1)$, independent of t, with $\mathrm{spr}[H(t)] < \alpha$ on all of T. We choose β with $\alpha < \beta < 1$, and represent $H(t)^n$ by the Dunford integral

$$H(t)^n = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=\beta} \zeta^n [\zeta I - H(t)]^{-1} \mathrm{d}\zeta.$$

It follows that $|H(t)^n| \le \frac{1}{2\pi}\oint_{|\zeta|=\beta} |\zeta^n|\,\left|[\zeta I - H(t)]^{-1}\right|\,|\mathrm{d}\zeta|$, so that K_0 exists with

$$|H(t)^n| \le K_0\,\beta^{n+1} \quad (\text{for all } n \ge 1,\ t \in T).$$

The derivative $(H(t)^n)'$ equals a sum of n terms $H(t)^{j-1}\,H'(t)\,H(t)^{n-j}$ $(1 \le j \le n)$. Applying the last upper bound for $|H(t)^n|$, it follows that there is a constant L_1 with $\left|(H(t)^n)'\right| \le L_1\, n\,\beta^n$ (for all $n \ge 1$, $t \in T$). Hence K_1, θ_1, with $0 < \theta_1 < 1$, exist such that

$$\left|\big(H(t)^n\big)'\right| \le K_1\,\theta_1^n \quad (\text{for all } n \ge 1,\ t \in T).$$

Differentiating the sum just mentioned, we can express $(H(t)^n)''$, in its turn, as a sum (of $3\,n$ terms). It follows that for some K_2, θ_2, with $0 < \theta_2 < 1$,

$$\left|\big(H(t)^n\big)''\right| \le K_2\,\theta_2^n \quad (\text{for all } n \ge 1,\ t \in T).$$

The lemma has thus been proved with

$$K = \max\{K_0, K_1, K_2\} \quad \text{and} \quad \theta = \max\{\beta, \theta_1, \theta_2\}. \qquad \square$$

Defining $G(t) = F_n^{[0]}(t)$, we see, in view of definition (5.11) and Remark 5.4, that

(5.13) $\qquad G(t)$ has a continuous second derivative on $[-\pi,\pi]$,

(5.14) $\qquad G(-\pi) = G(\pi),\ \ G'(-\pi) = G'(\pi).$

Therefore, by performing twice a partial integration, we get $\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}jt}\, G(t)\mathrm{d}t = \frac{-1}{2\pi\, j^2}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}jt}\, G''(t)\mathrm{d}t$ (for $j \ge 1$). Hence,

(5.15)
$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}jt}\, G(t)\mathrm{d}t\right| \le L_j, \quad L_0 = \max_t |G(t)|, \quad L_j = \frac{1}{j^2}\max_t |G''(t)| \ \ (\text{for } j \ge 1).$$

In order to bound $|G(t)|$ and $|G''(t)|$, we note that $G(t) = A(t) + B(t)$, with

$$A(t) = P(t)^n - [\phi(t)\,P(t)]^n \text{ and } B(t) = [\phi(t)F_0(t)]^n.$$

Clearly,

$$A(t) = 0 \quad (\text{for } |t| \le \varepsilon), \quad B(t) = 0 \quad (\text{for } 2\varepsilon \le |t| \le \pi).$$

In view of the material in Section 5.2, it follows that Lemma 5.5 can be applied with $H(t) = P(t)$ and $H(t) = \phi(t)P(t)$ on $T = \{t : \ \varepsilon \le |t| \le \pi\}$, as well as with $H(t) = \phi(t)F_0(t)$ on $T = \{t : \ |t| \le 2\varepsilon\}$. This leads, for some K, θ with $0 < \theta < 1$, to the bounds

$$A^{(p)}(t)| \le 2K\,\theta^n, \ \ |B^{(p)}(t)| \le K\,\theta^n \quad (\text{for } |t| \le \pi \text{ and } p = 0,1,2),$$
$$|G(t)| \le 3K\,\theta^n, \quad |G''(t)| \le 3K\,\theta^n \quad (\text{for } |t| \le \pi).$$

Using the last two upper bounds in combination with (5.15), we obtain

$$\sum_{j=0}^{\infty} |C_{nj}^{[0]}| = \sum_{j=0}^{\infty} \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}jt}\, G(t)\mathrm{d}t\Big| \le \sum_{j=0}^{\infty} L_j \le \left(1 + \sum_{j=1}^{\infty}\frac{1}{j^2}\right) 3K\theta^n.$$

Hence, putting $M_0 = \Big(1 + \sum_{j=1}^{\infty} \frac{1}{j^2}\Big)3K$, we have

$$\sum_{j=0}^{\infty} |C_{nj}^{[0]}| \le M_0 \quad (\text{for } n \ge 1). \tag{5.16}$$

Let $p+1 \le \ell \le p+q$. In order to bound $\big|C_{nj}^{[\ell]}\big| = \big|\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}jt}\, F_n^{[\ell]}(t)\mathrm{d}t\big|$, we put

$$G(t) = F_n^{[\ell]}(t),$$

and note that, because of definition (5.12) and Remark 5.4, we have again properties (5.13), (5.14), (5.15). Using that $G(t) = F_\ell(t)^n = (\lambda_\ell^*)^{n-1}F_\ell(t)$, we now have in (5.15) the equalities $L_0 = \max_t |F_\ell(t)|$ and $L_j = \frac{1}{j^2}\max_t |F_\ell''(t)|$ $(j \ge 1)$. Putting $M_\ell = \sum_{j=0}^{\infty} L_j$, we obtain

$$\sum_{j=0}^{\infty} |C_{nj}^{[\ell]}| \le M_\ell \quad (\text{for } p+1 \le \ell \le p+q \text{ and } n \ge 1). \tag{5.17}$$

5.4. **Part 4 of the proof: bounding $\sum_{j=0}^{\infty}\big|C_{nj}^{[\ell]}\big|$ for $1 \le \ell \le p$.** Three lemmas will be used; the first two are related to material in the seminal paper [41].

Lemma 5.6. *Let constants K, α be given. Suppose values δ_{nj} satisfy, for $n \ge 1$ and $j \ge 0$,*

$$0 \le \delta_{nj} \le \frac{K}{\sqrt{n}}, \tag{5.18}$$

$$0 \le \delta_{nj} \le \frac{K\,\sqrt{n}}{(j-\alpha\, n)^2} \quad (\textit{if } j \ne \alpha\, n). \tag{5.19}$$

Then a finite value M, depending only on K, exists with $\sum_{j=0}^{\infty} \delta_{nj} \le M$ (for all $n \ge 1$).

A version of this lemma was used, implicitly, in [41]. We omit the proof of Lemma 5.6, because it is simple and very similar to the proof of the related result given in [41], p. 278.

The following lemma is about $G_0(t), \mu(t)$, with values in $\mathbb{C}^{k\times k}$ and $\mathbb{C}$, respectively, where

$$G_0(t) \text{ and } \mu(t) \text{ are twice continuously differentiable on } [-\pi,\pi], \tag{5.20}$$

$$G_0(-\pi) = G_0(\pi),\ \ G_0'(-\pi) = G_0'(\pi) \ \text{ and } \ \mu(-\pi) = \mu(\pi),\ \ \mu'(-\pi) = \mu'(\pi), \tag{5.21}$$

$$|\mu(0)| = 1, \quad |\mu(t)| < 1 \text{ (for } 0 < |t| \le \pi), \tag{5.22}$$

$$\mu(t) = \mu(0)\cdot \exp[\alpha i t - (\beta + \delta \mathrm{i})t^2 + \mathcal{O}(t^3)] \text{ as } t \to 0, \text{ with real } \alpha,\beta,\delta, \text{ where } \beta > 0. \tag{5.23}$$

Lemma 5.7. *Assume $G_0(t), \mu(t)$ satisfy all of the conditions just mentioned, and let*

$$D_{nj} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}jt}\, \mu(t)^{n-1}\, G_0(t)\, dt, \quad \delta_{nj} = |D_{nj}|.$$

Then a constant K exists such that (5.18), (5.19) *hold (for all $n \ge 1$ and $j \ge 0$).*

Proof. The proof will be based on ideas taken from [41], pp. 277-278.

Proving (5.18). A combination of the fact that $|\mu(t)| < 1$ (for $0 < |t| \le \pi$) with the asymptotic expansion for $\mu(t)$ (when $t \to 0$) (see (5.22), (5.23)) shows that for some constant β_0, with $0 < \beta_0 < \beta$,

$$|\mu(t)| \le \mathrm{e}^{-\beta_0 t^2} \quad (\text{for } 0 \le |t| \le \pi). \tag{5.24}$$

Therefore,

$$\begin{aligned} |\delta_{nj}| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |\mu(t)|^{n-1}|G_0(t)|\mathrm{d}t \\ &\le \frac{\exp(\beta_0\pi^2)}{2\pi}\int_{-\pi}^{\pi} \exp(-\beta_0 n t^2)|G_0(t)|\mathrm{d}t \le \frac{K_0}{\sqrt{n}}\int_{-\infty}^{\infty} \exp(-\beta_0 x^2)\mathrm{d}x, \end{aligned}$$

for some constant K_0. Hence, a constant K exists as required in (5.18).

Proving (5.19). We assume $n \ge 1$, $\alpha n \ne j \ge 0$, and introduce the functions $H_0(t) = \mathrm{e}^{-\mathrm{i}\alpha t} G_0(t)$, $\nu(t) = \mathrm{e}^{-\mathrm{i}\alpha t}\mu(t)$, $H(t) = \nu(t)^{n-1} H_0(t)$, so that

$$D_{nj} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(j-\alpha n)t} H(t)\,\mathrm{d}t, \nu(t) = \nu(0)\cdot \exp[-(\beta+\delta\mathrm{i})t^2 + \mathcal{O}(t^3)] \text{ (for } t \to 0).$$

By two partial integrations, there follows $D_{nj} = \frac{-1}{2\pi(j-\alpha n)^2}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(j-\alpha n)t} H''(t)\,\mathrm{d}t$, so that

$$|D_{nj}| \le \frac{1}{(j-\alpha n)^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} |H''(t)|\,\mathrm{d}t.$$

We have $H''(t) = A_n(t) + B_n(t) + C_n(t)$, where

$$A_n(t) = (n-1)(n-2)\,[\nu'(t)]^2\,[\nu(t)]^{n-3}\,H_0(t),$$
$$B_n(t) = (n-1)[\nu(t)]^{n-2}\big[\nu''(t)H_0(t) + 2\,\nu'(t)H_0'(t)\big], \quad C_n(t) = [\nu(t)]^{n-1}H_0''(t).$$

Because (5.24) holds, and $\nu'(0) = 0$, we have for some constant K_0,

$$|\nu(t)| \le \mathrm{e}^{-\beta_0 t^2} \quad \text{and} \quad |\nu'(t)| \le K_0|t| \qquad (\text{for } 0 \le |t| \le \pi).$$

Combining these two upper bounds with the above expressions for $A_n(t), B_n(t)$, $C_n(t)$, we see that K_1, K_2, K_3 exist such that, for all $n \ge 1$,

$$\int_{-\pi}^{\pi} |A_n| \le K_1\, n^2 \int_{-\pi}^{\pi} |t|^2 \exp(-\beta_0 n t^2)\,\mathrm{d}t \le K_1\,\sqrt{n}\int_{-\infty}^{\infty} x^2 \exp(-\beta_0 x^2)\,\mathrm{d}x,$$
$$\int_{-\pi}^{\pi} |B_n| \le K_2\, n \int_{-\pi}^{\pi} \exp(-\beta_0 n t^2)\,\mathrm{d}t \le K_2\,\sqrt{n}\int_{-\infty}^{\infty} \exp(-\beta_0 x^2)\,\mathrm{d}x,$$
$$\int_{-\pi}^{\pi} |C_n| \le K_3.$$

It follows that there is a constant K with $|D_{nj}| \le \frac{K\sqrt{n}}{(j-\alpha n)^2}$ □

Lemma 5.8. *Let ℓ be given with $1 \le \ell \le p$. Then the function $F_n^{[\ell]}(t)$ (cf. definition* (5.12)*) can be written as $F_n^{[\ell]}(t) = \mu(t)^{n-1}G_0(t)$, with $\mu(t)$, $G_0(t)$ satisfying the assumptions* (5.20)–(5.23) *made in Lemma* 5.7.

Proof. From Section 5.2, we have $F_n^{[\ell]}(t) = \mu(t)^{n-1}G_0(t)$, where $\mu(t) = \phi(t)\,\lambda_\ell(t)$ and $G_0(t) = \phi(t)F_\ell(t)$ satisfy (5.20), (5.21), (5.22). To prove also (5.23), we note that

$$\mu(t) = \lambda_\ell(t) \quad (\text{for } |t| \le \varepsilon). \tag{5.25}$$

By Lemma 5.2, there are $\alpha_1, \alpha_2, \ldots$ (possibly depending on ℓ, but *not* on γ) such that

$$\lambda_\ell(t) = \lambda_\ell^*[1 + \alpha_1 z + \alpha_2 z^2 + \ldots] \;\; (\text{for } t \to 0), \;\text{with } z = -\gamma + \gamma\,\mathrm{e}^{-\mathrm{i}t} \text{ and } \alpha_1 > 0.$$

Defining real α, β, δ by $\alpha = -\alpha_1\gamma$ and $\beta + \delta\mathrm{i} = \frac{\gamma}{2}[\alpha_1 + (2\alpha_2 - \alpha_1^2)\gamma]$, we have

$$\lambda_\ell(t) = \lambda_\ell^* \cdot \exp[\alpha i t - (\beta + \delta\mathrm{i})t^2 + \mathcal{O}(t^3)] \quad (\text{for } t \to 0). \tag{5.26}$$

Clearly, $\beta = f(\gamma)$, with f defined by: $f(x) = \frac{x}{2}\left[\alpha_1 + \left(2\,\mathrm{Re}\{\alpha_2\} - \alpha_1^2\right)x\right]$ (for all real x).

Because $1 \ge |\lambda_\ell(t)/\lambda_\ell^*| = \exp[-\beta t^2 + \mathcal{O}(t^3)]$ (for $t \to 0$), there follows

$$f(\gamma) \ge 0.$$

Although this inequality has (formally) been derived only for the value γ at hand, it must evidently hold for *any* γ' with $0 < \gamma' < \gamma_0$. Therefore, also $f(\gamma_0) \ge 0$, which implies

$$\beta = f(\gamma) = \tfrac{\alpha_1\gamma}{2}(1 - \tfrac{\gamma}{\gamma_0}) + (\tfrac{\gamma}{\gamma_0})^2 f(\gamma_0) \ge \tfrac{\alpha_1\gamma}{2}(1 - \tfrac{\gamma}{\gamma_0}) > 0.$$

Combining the resulting inequality "$\beta > 0$" with (5.25), (5.26), we obtain (5.23) □

Because $C_{nj}^{[\ell]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}jt}\, F_n^{[\ell]}(t)\mathrm{d}t$, the above three lemmas imply that

$$\sum_{j=0}^{\infty} |C_{nj}^{[\ell]}| \le M_\ell \quad (\text{for } 1 \le \ell \le p \text{ and } n \ge 1), \tag{5.27}$$

for some constants M_ℓ not depending on n. The proof of Statement (I) in Theorem 2.1 is completed using (5.16), (5.17), (5.27) as indicated at the end of Section 5.1.

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