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Stability Estimates for Families of Matrices of Nonuniformly Bounded Order

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ABSTRACT

This paper addresses the problem of establishing upper bounds for the norm of the nth power of square matrices. The focus is on upper bounds that are valid for families of matrices of finite, but nonuniformly bounded, order. Such upper bounds are relevant to the stability analysis of numerical methods for solving differential equations. In the famous Kreiss matrix theorem a condition, on the resolvent of matrices, occurs which implies that the norm of the nth power does not grow faster than linearly with n. In this paper a slightly stronger version of this resolvent condition is studied, which is often satisfied in cases of practical interest. We prove that this version implies growth at a rate that is essentially lower than in the case of the classical Kreiss resolvent condition. We consider also a condition, on the iterated resolvent of matrices, which is related to one of the assumptions occurring in the Hille-Yosida theorem. This condition is known to imply growth at a rate $\mathscr{O}(n^{1/2})$. We prove that, when a slightly stronger version of the iterated resolvent condition is in force, there is growth at a rate that is at most $\mathscr{O}(n^p)$, with $0 \leqslant p < \frac{1}{2}$.

1. INTRODUCTION

1.1. Stability Estimates and Resolvent Conditions

In the stability analysis of numerical processes one is often faced with the problem of estimating the norm of the nth powers of given $s \times s$ matrices B. Stable numerical processes are distinguished by the property that the norm of

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B^n—as a function of n—does not grow at a faster rate than some power of n, say n^p. The case $p = 0$ corresponds to what sometimes is called *strong stability*, and $p > 0$ to *weak stability*; see e.g. Richtmyer and Morton (1967).

In this paper we deal with two conditions guaranteeing that the norm of B^n behaves as is required for stability. Both conditions will involve the so-called *resolvent* $(\zeta I - B)^{-1}$ of B. Here ζ is a complex variable, and I denotes the $s \times s$ identity matrix. In formulating our resolvent conditions we denote by $\|\cdot\|$ an arbitrary norm on the vector space $\mathbb{C}^{s,s}$ of all complex $s \times s$ matrices. Further, W stands for a subset of the *unit disk* $D = \{\zeta : \zeta \in \mathbb{C}$ with $|\zeta| \leqslant 1\}$, and $\mathrm{d}(\zeta, W) = \inf\{|\zeta - \xi| : \xi \in W\}$ denotes the *distance* from $\zeta \in \mathbb{C}$ to W.

Our first condition on the resolvent of B requires that a constant L exist such that

$$\zeta I - B \text{ is invertible, and } \left\|(\zeta I - B)^{-1}\right\| \leqslant L\mathrm{d}(\zeta, W)^{-1} \quad \text{for all } \zeta \in \mathbb{C} \setminus W. \tag{1.1}$$

We shall refer to this as the *Kreiss resolvent condition with respect to W, with constant L*. We use this terminology because one of the equivalent statements in the reputed Kreiss matrix theorem (see e.g. Richtmyer and Morton, 1967) is of the form (1.1) with W equal to the unit disk D.

The condition (1.1) is known to imply the stability estimate $\|B^n\| \leqslant eLs$ (for all $n \geqslant 1$); cf. Dorsselaer et al. (1993), Spijker (1991), Wegert and Trefethen (1994). The fact that the upper bound eLs depends on the order s is immaterial in the classical situation where the B stand for so-called amplification matrices (see Richtmyer and Morton, 1967). These B are the Fourier transforms of solution operators of finite difference methods and are of *fixed* finite order s. But, in important cases (approximations to initial–boundary value problems with variable coefficients or time discretizations to spectral methods) the matrices B stand for the numerical solution operators themselves. In such cases we have to deal with a family of matrices B of finite but *nonuniformly* bounded order s. In fact, the order of these matrices has to increase without bound if the error in the numerical approximations is to approach zero. In such cases it would be desirable to have an upper bound for $\|B^n\|$ that is independent of s. Accordingly, in this paper we focus, for given L and W, on upper bounds that may depend only on n and are valid *uniformly* for all $B \in \mathbb{C}^{s,s}$, $s \geqslant 1$, satisfying (1.1).

Under the condition (1.1), we have the integral representation

$$B^n = \frac{1}{2\pi i} \int_\Gamma \zeta^n (\zeta I - B)^{-1} \, \mathrm{d}\zeta,$$

where Γ is the positively oriented circle $|\zeta| = 1 + 1/n$ (see e.g. Dowson, 1978). Using the inequality from (1.1), with W equal to the unit disk, the norm of the above integral can easily be estimated so as to get

$$\|B^n\| \leqslant 4Ln \qquad (\text{for } n \geqslant 1).$$

The exponent $p = 1$ of this weak stability estimate is essentially best possible when the general condition (1.1) is in force with $W = D$; see Dorsselaer et al. (1993), Kraaijevanger (1994). The question poses itself of whether weak stability with $p < 1$, or even strong stability, can be proved under slightly stronger conditions on B [a similar question was also raised in Gorelick and Kranzer (1976), Tadmor (1986)].

In the present paper we address this question. We shall deal with a version of (1.1) which is stronger than the classical resolvent condition, as occurring in the Kreiss matrix theorem, in that W is strictly contained in the unit disk (i.e., $W \subset D$ and $W \neq D$). In view of applications of stability estimates to actual numerical processes, we still allow W to have a finite number of points in common with the unit circle $|\zeta| = 1$.

The second resolvent condition which we deal with in this paper reads

$$\begin{aligned}&\zeta I - B \text{ is invertible, and } \|(\zeta I - B)^{-k}\| \leqslant L\mathrm{d}(\zeta, W)^{-k}\\ &\text{whenever } \zeta \in \mathbb{C} \setminus W \text{ and } k \geqslant 1.\end{aligned} \tag{1.2}$$

We shall refer to this as the *Hille-Yosida resolvent condition with respect to* W, *with constant* L. The reason for this terminology lies in the fact that one of the equivalent statements in the famous Hille-Yosida theorem (see e.g. Pazy, 1983, p. 20) can be cast into the form (1.2), with W equal to a half plane $\{\zeta : \zeta \in \mathbb{C} \text{ with } \operatorname{Re} \zeta \leqslant \omega\}$.

The condition (1.2), with W equal to the unit disk D, implies the stability estimate

$$\|B^n\| \leqslant eLn^{1/2} \qquad (\text{for } n \geqslant 1).$$

This follows from the material in Bonsall and Duncan (1980, p. 41) in combination with Lenferink and Spijker (1990), or from Lubich and Nevanlinna (1991). The exponent $p = \frac{1}{2}$ in this estimate is essentially sharp; see McCarthy (1993), Dorsselaer et al. (1993).

The question arises of whether estimates with $p < \frac{1}{2}$ are valid when the set W occurring in the resolvent condition (1.2) is strictly contained in the unit disk (and still contains points with modulus $|\zeta| = 1$). This question also will be studied in this paper.

1.2. Contents of the Paper

In Section 2.1 we study a few geometric properties of certain sets W that are said to be of *type* q, where $0 \leqslant q < \infty$. These sets are strictly contained in the unit disk D, and the contact between these sets and the unit circle is of order q. In Section 2.2 we elucidate the Kreiss resolvent condition with respect to such sets of type q, and we make some preparations that will be useful in deriving stability estimates from the Kreiss condition. Similarly, in Section 2.3 we elucidate the Hille-Yosida condition with respect to sets of type q.

In Section 3.1 we prove stability estimates of the form

$$\|B^n\| \leqslant \gamma L n^p \qquad (\text{for } n \geqslant 1) \tag{1.3}$$

under the Kreiss condition with respect to sets W of type q. Here γ depends only on W and q, whereas the exponent p is given by

$$p = 1 - \frac{1}{1+q} \qquad \text{for} \quad 0 \leqslant q < \infty. \tag{1.4}$$

In Section 3.2 we present a variant of these stability estimates for the case where the upper bound $Ld(\zeta, W)^{-1}$ in (1.1) is replaced by $Ld(\zeta, W)^{-1-\alpha}$, with $\alpha > 0$. Moreover, we give a counterexample which shows that, in general, the exponent given by (1.4) is sharp.

In Section 4 we derive estimates of the form (1.3) under the Hille-Yosida condition with W of type q. Again γ depends only on W and q, but now

$$p = \max\left\{0, \frac{1}{2} - \frac{1}{1+q}\right\} \qquad \text{for} \quad 0 \leqslant q < \infty. \tag{1.5}$$

Section 5 contains two technical lemmas that are used in some proofs in Sections 2 and 3.

A concise illustration of the estimates of the Sections 3, 4 is obtained by choosing W equal to the closed disk with center ξ and radius $1 - |\xi|$, where $\xi \in \mathbb{C}$, $0 < |\xi| < 1$. This set W is of type q with $q = 1$. In view of (1.4), the Kreiss condition (1.1) implies the estimate

$$\|B^n\| \leqslant \gamma L n^{1/2} \qquad (\text{for } n \geqslant 1),$$

whereas, by (1.5), the Hille-Yosida condition (1.2) implies strong stability:

$$\|B^n\| \leqslant \gamma L \qquad (\text{for } n \geqslant 1).$$

In these two estimates γ only depends on ξ (and not on s, n, L, or the norm $\|\cdot\|$).

We note that some of the conclusions at which we arrive in this paper apply equally well to the elements of Banach algebras, instead of $s \times s$ matrices B. We have formulated our material, however, in terms of matrices, since applications of the estimates established in this paper are expected to lay mainly in the latter field. In Spijker and Straetemans (1995) our estimates will be applied to actual numerical processes for solving initial(–boundary) value problems in ordinary and partial differential equations.

2. RESOLVENT CONDITIONS WITH RESPECT TO SUBSETS OF THE UNIT DISK

2.1. *Subsets of the Unit Disk of Type q*

Let $\xi_1, \xi_2, \dots, \xi_m$ be complex numbers with modulus $|\xi_j| = 1$ for $1 \leqslant j \leqslant m$, and let the unit disk again be denoted by

$$D = \{\zeta : \zeta \in \mathbb{C} \text{ with } |\zeta| \leqslant 1\}.$$

We assume

$$W \text{ is a closed subset of } D; \tag{2.1a}$$

$$\text{The intersection of } W \text{ with the unit circle } |\zeta| = 1 \text{ consists of the points } \xi_1, \xi_2, \dots, \xi_m. \tag{2.1b}$$

Let $q \geqslant 0$ be a given real constant. We consider the situation where the distance between the points $\xi \in W$ and the unit circle is not smaller than a multiple of $|\xi - \xi_j|^{1+q}$, for ξ close to ξ_j, i.e.,

$$\text{There exist positive } \beta_0, \beta_1 \text{ such that } 1 - |\xi| \geqslant \beta_1 |\xi - \xi_j|^{1+q} \text{ whenever } \xi \in W,\ 1 \leqslant j \leqslant m,\ |\xi - \xi_j| \leqslant \beta_0. \tag{2.2}$$

DEFINITION 2.1. A set W is said to be of *type* q if the conditions (2.1), (2.2) are fulfilled.

Below we shall present two useful conditions that are equivalent to (2.2). In order to formulate these conditions we need some notation, which we introduce first.

For complex $\zeta \neq 0$ we choose $\operatorname{Arg}(\zeta)$, the principal value of the argument, in the interval $(-\pi, \pi]$, and we put $\operatorname{Arg}(0) = 0$.

Further, we define $\xi_0 = \xi_m$ and, without loss of generality, assume that

$$\xi_{j+1} = \xi_j \exp(2i\sigma_j) \quad \text{with } \sigma_j > 0 \qquad (j = 0, 1, \dots, m-1),$$

where

$$\sigma_0 + \sigma_1 + \cdots + \sigma_{m-1} = \pi.$$

We put $\sigma_m = \sigma_0$ and introduce, for $1 \leqslant j \leqslant m$, the sectors

$$S_j = \left\{ \zeta : \zeta = |\zeta| \cdot \xi_j e^{i\theta} \text{ with } -\sigma_{j-1} \leqslant \theta \leqslant \sigma_j \right\}.$$

We define subsets W_δ of the unit disk by

$$W_\delta = \bigcup_{j=1}^{m} \left\{ \xi : \xi \in S_j \;\&\; |\xi| \leqslant 1 - \delta \left| \operatorname{Arg}\left(\xi \xi_j^{-1} \right) \right|^{1+q} \right\}, \tag{2.3a}$$

where we restrict δ to the values

$$0 < \delta < \pi^{-1-q}. \tag{2.3b}$$

We shall deal with the following two conditions on the set W:

There exist positive γ_0, γ_1 such that $|\xi| \leqslant 1 - \gamma_1 |\operatorname{Arg}(\xi\xi_j^{-1})|^{1+q}$ whenever $\xi \in W$, $1 \leqslant j \leqslant m$, $|\operatorname{Arg}(\xi\xi_j^{-1})| \leqslant \gamma_0$.

(2.4)

There is a set W_δ of the form (2.3) such that $W \subset W_\delta$. (2.5)

THEOREM 2.2. *Assume* (2.1). *Then the three conditions* (2.2), (2.4), *and* (2.5) *are equivalent to each other.*

Each of the three equivalent conditions (2.2), (2.4), (2.5) has some advantages over the other two: The condition (2.2) has a quite simple structure in that only distances (and no arguments) in the complex plane are involved; further, (2.2) is useful in certain applications of the results of this paper (see Spijker and Straetemans, 1995). The condition (2.4) allows of a transparent geometrical interpretation; it says that the *contact* of W and the unit circle is of *order* q. The condition (2.5) will be useful in deriving our stability estimates (in Sections 3, 4).

Proof of Theorem 2.2.

(1) Assume (2.2). Choose $\gamma_0 \in (0, \pi/2)$ so small that, for $1 \leqslant j \leqslant m$, the sets

$$H_j = \left\{ \zeta : 1 - \gamma_0 \leqslant |\zeta| \leqslant 1 \;\&\; \left| \operatorname{Arg}\left(\zeta \xi_j^{-1} \right) \right| \leqslant \gamma_0 \right\}$$

lie within the corresponding disks $\{\zeta : |\zeta - \xi_j| \leqslant \beta_0\}$. Let $\xi \in W \cap H_j$ and $\theta = \operatorname{Arg}(\xi \xi_j^{-1})$. Then

$$|\xi| \leqslant 1 - \beta_1 |\xi - \xi_j|^{1+q},$$

and

$$|\xi - \xi_j| = \left|1 - |\xi| e^{i\theta}\right| \geqslant \min\left\{ |1 - \rho e^{i\theta}| : 0 \leqslant \rho \leqslant 1 \right\} = |\sin \theta| \geqslant \frac{2}{\pi} |\theta|,$$

so that

$$|\xi| \leqslant 1 - \beta_1 \left| \frac{2}{\pi} \operatorname{Arg}\left(\xi \xi_j^{-1} \right) \right|^{1+q}.$$

Consequently, (2.4) holds with $\gamma_1 = \min[\beta_1 (2/\pi)^{1+q}, \gamma_0^{-q}]$.

(2) Assume (2.4). Since the sets

$$T_j = \left\{ \xi : \xi \in W \cap S_j \text{ with } \left| \operatorname{Arg}\left(\xi \xi_j^{-1} \right) \right| \geqslant \gamma_0 \right\}$$

are compact, the quantity

$$\epsilon = \inf\left\{ |\zeta - \xi| : |\zeta| = 1 \;\&\; \xi \in T_1 \cup T_2 \cup \cdots \cup T_m \right\}$$

is positive. We see that $W \subset W_\delta$ with $\delta = \min\{\gamma_1, \epsilon \pi^{-1-q}, 2^{-1} \pi^{-1-q}\}$, which proves (2.5).

(3) Assume (2.5). We shall complete the proof by establishing (2.2). Let $\xi = |\xi| \xi_j e^{i\theta}$ with $\theta = \operatorname{Arg}(\xi \xi_j^{-1})$. By Taylor series expansion around $\theta = 0$ one easily sees that

$$|\xi - \xi_j|^2 = (1 - |\xi|)^2 + |\xi| \theta^2 + \mathscr{O}(\theta^4)$$

uniformly for $|\xi| \leqslant 1$, $1 \leqslant j \leqslant m$. In view of (2.5) we can choose $\beta_0 > 0$ so small that, for

$$\xi \in W, \qquad 1 \leqslant j \leqslant m, \qquad |\xi - \xi_j| \leqslant \beta_0,$$

the inequality $\delta|\theta|^{1+q} \leqslant 1 - |\xi|$ holds. This inequality implies

$$\theta^2 \leqslant \delta^{-\alpha}(1 - |\xi|)^{\alpha} \qquad \text{with} \quad \alpha = \frac{2}{1+q}.$$

We insert this upper bound for θ^2 in the above expression for $|\xi - \xi_j|^2$, and we raise both members of the resulting inequality to the power $(1 + q)/2$. This yields

$$|\xi - \xi_j|^{1+q} \leqslant \left\{(1 - |\xi|)^2 + \mathscr{O}\left[(1 - |\xi|)^{\alpha}\right]\right\}^{1/\alpha} \leqslant (\beta_1)^{-1}(1 - |\xi|),$$

where β_1 is a positive constant as required in (2.2). ■

2.2. *Interpretations of the Kreiss Resolvent Condition with Respect to Sets of Type q*

Let B be an arbitrary $s \times s$ matrix, and let $\|\cdot\|$ be a norm on $\mathbb{C}^{s,s}$. In elucidating the Kreiss resolvent condition (1.1) we shall make use of the following definition [cf. Reddy and Trefethen (1990, 1992), Trefethen (1995), Dorsselaer et al. (1993)].

DEFINITION 2.3. The *ϵ-pseudospectrum* of B is the set of all $\lambda \in \mathbb{C}$ for which an $E \in \mathbb{C}^{s,s}$, with $\|E\| \leqslant \epsilon$, exists such that λ is an eigenvalue of $B + E$.

The following three conditions on the matrix B will be related to each other in the subsequent theorem:

W is a set of type q, and L a constant such that the Kreiss resolvent condition (1.1) is fulfilled; (2.6)

$\epsilon_0 > 0$, $L_0 > 0$, and W_δ [see (2.3)] are such that $\zeta I - B$ is regular for all $\zeta \in \mathbb{C} \setminus W_\delta$, and $\|(\zeta I - B)^{-1}\| \leqslant L_0 \mathrm{d}(\zeta, W_\delta)^{-1}$ for all ζ with $0 < \mathrm{d}(\zeta, W_\delta) \leqslant \epsilon_0$; (2.7)

W is a set of type q, and L a constant such that, for each $\epsilon > 0$, the ϵ-pseudospectrum of B is contained in the set $\{\zeta : \zeta = \xi + \eta \text{ with } \xi \in W,\ \eta \in \mathbb{C}, \text{ and } |\eta| \leqslant L\epsilon\}$. (2.8)

THEOREM 2.4. *Let ξ_j and q be as in Section 2.1, and $B \in \mathbb{C}^{s,s}$.*

(a) *Let $\|\cdot\|$ be an arbitrary norm on the vector space $\mathbb{C}^{s,s}$. Then (2.6) implies (2.7) with $\epsilon_0 = \infty$, $L_0 = L$, and δ only depending on W and q. Conversely, (2.7) implies (2.6) with $W = W_\delta$ and $L = c_0 L_0$, where c_0 only depends on ϵ_0, W_δ, and q.*

(b) *Let $\|\cdot\|$ be a matrix norm induced by a vector norm on $\mathbb{C}^s$. Then (2.6) and (2.8) are equivalent.*

This theorem provides us with two new conditions equivalent to the Kreiss condition when W is of type q. The theorem is closely related to material presented in Reddy and Trefethen (1992). The condition (2.8) allows of a straightforward intuitive interpretation, whereas (2.7) is very useful in proving stability estimates [see Section 3.1 of the present paper and Spijker and Straetemans (1995)].

Proof of Theorem 2.4.

(1) Part (b) of the theorem is an easy consequence of the fact that, when $\|\cdot\|$ is induced by a vector norm on $\mathbb{C}^s$, λ belongs to the ϵ-pseudospectrum if and only if

$$\lambda \text{ is an eigenvalue of } B \quad \text{or} \quad \|(\lambda I - B)^{-1}\| \geqslant \epsilon^{-1}$$

[see Reddy and Trefethen (1990, 1992) or Dorsselaer et al. (1993)].

(2) Assume (2.6). In view of Theorem 2.2, there is a set W_δ (only depending on W and q) with $W \subset W_\delta$. For $\zeta \in \mathbb{C} \setminus W_\delta$ we have $\mathrm{d}(\zeta, W)^{-1} \leqslant \mathrm{d}(\zeta, W_\delta)^{-1}$, so that (2.7) holds with $\epsilon_0 = \infty$, $L_0 = L$.

(3) Assume (2.7), and put $W = W_\delta$. We shall prove (2.6) by using the integral representation

$$(zI - B)^{-1} = \frac{1}{2\pi i}\int_\Gamma (z - \zeta)^{-1}(\zeta I - B)^{-1}\,\mathrm{d}\zeta$$

$$(\text{for } z \in \mathbb{C} \text{ with } \mathrm{d}(z, W) > \epsilon_0),$$

where $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_m$ is the positively oriented curve composed of the segments $\Gamma_j = \{\zeta : \zeta \in S_j \text{ with } |\zeta| = 1 - \delta|\mathrm{Arg}(\zeta\xi_j^{-1})|^{1+q} + \epsilon_0/2\}$. In view of (2.7) and Lemma 5.1 we have

$$\|(\zeta I - B)^{-1}\| \leqslant L_0 \mathrm{d}(\zeta, W)^{-1} \leqslant 2L_0(c\epsilon_0)^{-1} \qquad \text{for} \quad \zeta \in \Gamma,$$

where c is specified in Lemma 5.1. Taking norms in our integral representation, we arrive at

$$\|(zI - B)^{-1}\| \leqslant (\pi c \epsilon_0)^{-1} L_0 \int_\Gamma |z - \zeta|^{-1} |d\zeta| \qquad \text{for} \quad \mathrm{d}(z, W) > \epsilon_0.$$

Let $\mathrm{d}(z, W) > \epsilon_0$ and $\zeta \in \Gamma$. Choosing $\xi \in W$ with $|\xi - \zeta| = \epsilon_0/2$, we obtain $|z - \zeta| \geqslant |z - \xi| - |\xi - \zeta| \geqslant \mathrm{d}(z, W) - \epsilon_0/2 \geqslant \frac{1}{2}\mathrm{d}(z, W)$, so that

$$|z - \zeta|^{-1} \leqslant 2\mathrm{d}(z, W)^{-1}.$$

Inserting this inequality in the above upper bound for $\|(zI - B)^{-1}\|$, there follows

$$\|(zI - B)^{-1}\| \leqslant c_1 L_0 \mathrm{d}(z, W)^{-1} \qquad \text{for} \quad \mathrm{d}(z, W) > \epsilon_0,$$

where c_1 equals the product of $2(\pi c \epsilon_0)^{-1}$ and the length of Γ. Consequently, (2.6) holds with $L = c_0 L_0$, $c_0 = \max\{1, c_1\}$. ■

2.3. Interpretations of the Hille-Yosida Resolvent Condition with Respect to Sets of Type q

We start with some definitions that will be used in the subsequent to elucidate the Hille-Yosida condition.

For $V \subset \mathbb{C}$ we denote by ∂V the *boundary* of V, and by $\mathrm{conv}(V)$ the *convex hull* (i.e. the intersection of all convex sets containing V).

Let ξ belong to the boundary ∂V of an arbitrary convex set $V \subset \mathbb{C}$. If ω is a real constant such that

$$\mathrm{Re}\{e^{-i\omega}(\zeta - \xi)\} \leqslant 0 \qquad \text{for all} \quad \zeta \in V,$$

then ω is called a *normal direction* to V at ξ.

For given $s \times s$ matrices B and norms $\|\cdot\|$ on $\mathbb{C}^{s,s}$, we shall use the following definition (cf. Dorsselaer et al., 1993; Lenferink and Spijker, 1990; Spijker, 1993).

DEFINITION 2.5. The *L-numerical range* of B, denoted by $\tau[B, L]$, is equal to the intersection of all disks $\{\zeta : \zeta \in \mathbb{C}, |\zeta - \gamma| \leqslant \rho\}$, where $\gamma \in \mathbb{C}$, $\rho \geqslant 0$ are such that

$$\|(B - \gamma I)^k\| \leqslant L\rho^k \qquad \text{for} \quad k = 1, 2, 3, \dots .$$

We deal with the following three conditions on $s \times s$ matrices B:

W is a set of type q, and L is a constant such that B satisfies the Hille-Yosida resolvent condition (1.2). (2.9)

The constant L and the set W_δ [see (2.3)] are such that, for all $\zeta \in \mathbb{C}$, $\|\exp[\zeta B]\| \leqslant L \exp[\mathrm{Re}(\xi\zeta)]$ whenever ξ is a boundary point of $V = \mathrm{conv}(W_\delta)$ at which $\omega = -\mathrm{Arg}(\zeta)$ is a normal direction. (2.10)

V is a set of type q, and L is a constant such that $\tau[B, L] \subset V$. (2.11)

THEOREM 2.6. *Let ξ_j and q be as in Section* 2.1, *and $B \in \mathbb{C}^{s,s}$.*

(a) *Let $\|\cdot\|$ be an arbitrary norm on the vector space $\mathbb{C}^{s,s}$. Then* (2.9) *implies* (2.10) *with δ depending only on W and q. Conversely,* (2.10) *implies* (2.9) *with $W = \mathrm{conv}(W_\delta)$.*

(b) *Let $\|\cdot\|$ be a norm on the vector space $\mathbb{C}^{s,s}$ which is submultiplicative (i.e., $\|A_1 A_2\| \leqslant \|A_1\| \|A_2\|$ for all $A_1, A_2 \in \mathbb{C}^{s,s}$). Then* (2.9) *implies* (2.11) *with $V = \mathrm{conv}(W)$. Conversely, if the norm $\|\cdot\|$ is submultiplicative and $\|I\| \leqslant L$, then* (2.11) *implies* (2.9) *with $W = V$.*

This theorem is closely related to material presented in Lenferink and Spijker (1990) and Spijker (1993). According to the theorem, the Hille-Yosida condition (1.2), with W of type q, can be interpreted as a bound on the exponential function $\exp[\zeta B]$ [see (2.10)], or as an inclusion for the L-numerical range [see (2.11)]. The interpretation (2.10) will be used in Section 4.

Proof of Theorem 2.6.

(1) We use the notation

$$B(\xi, \omega) = e^{-i\omega}(B - \xi I).$$

Assume (2.9). By Theorem 2.2 there is a set W_δ, depending only on W and q, such that B satisfies the Hille-Yosida condition with respect to W_δ. Hence, (1.2) also holds with W replaced by $V = \mathrm{conv}(W_\delta)$. This version of (1.2) is equivalent to

$\lambda I - B(\xi, \omega)$ is invertible, and $\|[\lambda I - B(\xi, \omega)]^{-k}\| \leqslant L\lambda^{-k}$ (for $k = 1, 2, 3, \ldots$) whenever $\lambda > 0$, $\xi \in \partial V$, and ω is a normal direction to V at ξ. (2.12)

For any given $\zeta \neq 0$ we choose $\xi \in \partial V$ at which $\omega = -\mathrm{Arg}(\zeta)$ is a normal direction. We have

$$\exp[-\mathrm{Re}(\xi\zeta)]\|\exp[\zeta B]\| = \|\exp[|\zeta|B(\xi, \omega)]\|$$
$$= \lim_{k \to \infty} \left\| [I - k^{-1}|\zeta|B(\xi, \omega)]^{-k} \right\|.$$

From (2.12) we see that this limit does not exceed L. This proves (2.10).

(2) Assume (2.10). Then

$$\|\exp[\lambda B(\xi, \omega)]\| = \|\exp[|\zeta|B(\xi, \omega)]\| \leqslant L$$

whenever $\zeta = \lambda e^{-i\omega}$, $\lambda > 0$, and ω is a normal direction to V at $\xi \in \partial V$. Using the formula

$$[\lambda I - B(\xi, \omega)]^{-k} = \frac{1}{(k-1)!}\int_0^\infty t^{k-1}e^{-\lambda t}e^{tB(\xi, \omega)}\,dt,$$

we see that (2.12) holds with $V = \mathrm{conv}(W_\delta)$. Therefore, (1.2) is satisfied, with W replaced by V. Since $\mathrm{conv}(W_\delta)$ is of type q, the matrix B satisfies (2.9) with $W = \mathrm{conv}(W_\delta)$.

(3) Let the norm $\|\cdot\|$ be submultiplicative, and assume (2.9). The matrix B satisfies the Hille-Yosida condition with constant L with respect to the set $V = \mathrm{conv}(W)$. By letting $|\zeta| \to \infty$ in this condition, we see that $L \geqslant \|I\|$. Theorem 2.1 of Lenferink and Spijker (1990) tells us that a Hille-Yosida condition, with respect to a convex set V and with a constant $L \geqslant \|I\|$, implies $\tau[B, L] \subset V$. Since $\mathrm{conv}(W)$ is of type q, the condition (2.11) is fulfilled.

(4) Let the norm $\|\cdot\|$ be submultiplicative, and assume $L \geqslant \|I\|$. According to Theorem 2.1 in Lenferink and Spijker (1990), the matrix B satisfies the Hille-Yosida condition with constant L with respect to the set $\tau[B, L]$. Therefore, (2.11) implies (2.9) with $W = V$. ∎

3. STABILITY ESTIMATES UNDER THE KREISS RESOLVENT CONDITION

3.1. *Conditions for Weak and Strong Stability*

The following theorem gives stability estimates under the Kreiss resolvent condition. The proof relies on Theorem 2.4, Lemma 5.1, and integral representations for B^n that have similarity to those used e.g. by Palencia and

García-Archilla (1993), Lenferink and Spijker (1991), and Lubich and Nevanlinna (1991).

THEOREM 3.1. *Let ξ_j and q be as in Section 2.1, and W of type q. Then there is a constant γ, depending only on W and q, such that*

$$\|B^n\| \leqslant \gamma L n^p \quad (\text{for } n \geqslant 1), \qquad \text{with} \quad p = 1 - \frac{1}{1+q},$$

whenever $L > 0$, $s \geqslant 1$, $\|\cdot\|$ is a norm on the vector space $\mathbb{C}^{s,s}$, and the $s \times s$ matrix B satisfies (1.1).

Proof.

(1) Assume (1.1). In view of Theorem 2.4, there is a set W_δ, depending only on W and q, such that (1.1) also holds with W replaced by W_δ. We write

$$B^n = \frac{1}{2\pi i}\int_\Gamma \zeta^n(\zeta I - B)^{-1}\,\mathrm{d}\zeta,$$

where Γ is a positively oriented Jordan curve with W_δ in its interior.

(2) Assume first $m = 1$, $\xi_1 = 1$. We choose Γ symmetric with respect to the real axis; the section of Γ with nonnegative imaginary part consists of the curves Δ_0, Δ_1 given by

$$\Delta_0 : \zeta = (1 - \delta\theta^{1+q} + \epsilon)e^{i\theta} \text{ for } 0 \leqslant \theta \leqslant \theta_0,$$

$$\Delta_1 : \zeta = (1 - \delta\theta^{1+q}/2)e^{i\theta} \text{ for } \theta_0 \leqslant \theta \leqslant \pi.$$

Here $\theta_0 \in (0, \pi)$ will be specified below, and $\epsilon = (\delta/2)\theta_0^{1+q}$. In view of the symmetry of Γ the above integral representation yields

$$\|B^n\| \leqslant \pi^{-1}L(I_0 + I_1),$$

$$\text{with} \quad I_j = \int_{\Delta_j} |\zeta|^n\, \mathrm{d}(\zeta, W_\delta)^{-1}|\mathrm{d}\zeta| \quad (j = 0, 1).$$

Along Δ_j we have $|\mathrm{d}\zeta| \leqslant c_0|\mathrm{d}\theta|$, with

$$c_0 = \left[\left(\frac{\delta(1+q)\pi^q}{2}\right)^2 + \left(1 + \frac{\delta\pi^{1+q}}{2}\right)^2\right]^{1/2}.$$

Applying Lemma 5.1, we thus obtain

$$I_0 \leqslant c_1 \int_0^{\theta_0} \left(1 - \delta\theta^{1+q} + \frac{\delta\theta_0^{1+q}}{2}\right)^n \theta_0^{-1-q}\, \mathrm{d}\theta,$$

$$I_1 \leqslant c_1 \int_{\theta_0}^{\pi} \left(1 - \frac{\delta\theta^{1+q}}{2}\right)^n \theta^{-1-q}\, \mathrm{d}\theta,$$

where $c_1 = 2c_0(c\delta)^{-1}$ depends only on δ and q. Choosing $\theta_0 = n^{p-1}$, we obtain

$$I_0 \leqslant c_1 \int_0^{\theta_0} \exp\left(\frac{\delta}{2}\right) n\, \mathrm{d}\theta = c_1 n^p \exp\left(\frac{\delta}{2}\right),$$

and

$$I_1 \leqslant c_1 \int_{\theta_0}^{\pi} \exp\left(-\frac{\delta\theta^{1+q}n}{2}\right) \theta^{-1-q}\, \mathrm{d}\theta \leqslant c_1 n^p \int_1^{\infty} \exp\left(-\frac{\delta x^{1+q}}{2}\right) x^{-1-q}\, \mathrm{d}x.$$

Hence $I_0 + I_1 \leqslant \gamma_0 n^p$, with γ_0 depending only on δ and q. Combining this inequality with the above upper bound for $\|B^n\|$, we obtain $\|B^n\| \leqslant \gamma L n^p$, with γ as required.

(3) It is clear that the stability estimate just proved for $m = 1$, $\xi_1 = 1$ remains valid for $m = 1$ and general ξ_1 with $|\xi_1| = 1$. In the general case, $m \geqslant 2$, the parts of Γ lying in the sectors S_j (see Section 2.1) equal $\Gamma_{0,j} + \Gamma_{1,j}$ with

$$\Gamma_{0,j} : \zeta = (1 - \delta|\theta|^{1+q} + \epsilon)\xi_j e^{i\theta} \text{ if } -\sigma_{j-1} \leqslant \theta \leqslant \sigma_j, |\theta| \leqslant \theta_0,$$

$$\Gamma_{1,j} : \zeta = (1 - \delta|\theta|^{1+q}/2)\xi_j e^{i\theta} \text{ if } -\sigma_{j-1} \leqslant \theta \leqslant \sigma_j, |\theta| \geqslant \theta_0.$$

Here $1 \leqslant j \leqslant m$, $\epsilon = \delta\theta_0^{1+q}/2$, and $\theta_0 = n^{p-1}$. Our integral representation for B^n yields

$$\|B^n\| \leqslant \pi^{-1} L \sum_{j=1}^{m} (I_{0,j} + I_{1,j}),$$

where, as in part (2) of the proof, $I_{0,j}$ and $I_{1,j}$ are integrals of $|\zeta|^n \mathrm{d}(\zeta, W_\delta)^{-1}$ along the parts of $\Gamma_{0,j}$ and $\Gamma_{1,j}$ where the local parameter θ is nonnegative.

Invoking again Lemma 5.1, in order to bound $d(\zeta, W_\delta)^{-1}$, we arrive at $I_{0,j} + I_{1,j} \leqslant \gamma_0 n^p$ with the same constant γ_0 as in part 2. We thus have $\|B^n\| \leqslant \gamma L n^p$, with $\gamma = \pi^{-1} m \gamma_0$. ■

3.2. A Generalization and a Counterexample

As in Friedland (1981), Reddy and Trefethen (1992), and Tadmor (1981), one may consider, for $\alpha > 0$, the following variant of the Kreiss resolvent condition:

$$\zeta I - B \text{ is invertible, and } \|(\zeta I - B)^{-1}\| \leqslant L d(\zeta, W)^{-1-\alpha} \text{ for all } \zeta \in \mathbb{C} \setminus W. \tag{3.1}$$

THEOREM 3.2. *Let ξ_j and q be as in Section* 2.1, *and W of type q. Let $\alpha > 0$, and*

$$p = 1 + \alpha - \frac{1}{1+q}.$$

Then there is a constant γ, depending only on W, q, and α, such that (1.3) *holds whenever $L > 0$, $s \geqslant 1$, $\|\cdot\|$ is a norm on the vector space $\mathbb{C}^{s,s}$, and the $s \times s$ matrix B satisfies* (3.1).

This theorem yields a generalization of Theorem 3.1 to the case of the resolvent condition (3.1). We omit the proof, since it consists in a straightforward modification of the above proof in Section 3.1.

Kraaijevanger (1994) gave a very interesting counterexample involving matrices B, with $\|B^n\| \geqslant cn$ and fixed $c > 0$, under the resolvent condition (1.1) with $W = D$. The following theorem is a nontrivial extension of Kraaijevanger's result to sets W of type q. It shows that the exponent p given in Theorem 3.1 is sharp.

THEOREM 3.3. *Let $q \geqslant 0$ be given, and define, for all $n \times n$ matrices $A = (\alpha_{jk})$, the norm of A by*

$$\|A\| = \max\{|\alpha_{j1}| + |\alpha_{j2}| + \cdots + |\alpha_{jn}| : j = 1, 2, \ldots, n\}.$$

Then constants $L > 0$, $c > 0$ and a set W of type q exist such that, for each $n \geqslant 1$, there is an $n \times n$ matrix B satisfying both (1.1) *and the inequality*

$$\|B^n\| \geqslant cn^p \qquad \text{with} \quad p = 1 - \frac{1}{1+q}.$$

Proof.

(1) Let $n \geqslant 1$ be given. Following Kraaijevanger (1994), we define $n \times n$ matrices $P_1, P_2, \dots, P_n$ as follows:

$$P_1 = (\alpha_{ij}) \quad \text{with} \quad \alpha_{i1} = 1 \text{ for } 1 \leqslant i \leqslant n, \text{ and } \alpha_{ij} = 0 \text{ otherwise.}$$

Further, for $2 \leqslant k \leqslant n$,

$$P_k = (\alpha_{ij})$$

with

$$\alpha_{i,k-1} = -1, \ \alpha_{i,k} = 1 \quad \text{for } k \leqslant i \leqslant n, \quad \text{and} \quad \alpha_{ij} = 0 \quad \text{otherwise.}$$

One easily verifies that

$$P_j P_k = O \ (j \neq k), \quad P_j P_j = P_j, \qquad P_1 + P_2 + \cdots + P_n = I.$$

We choose a fixed set W according to (5.1a) with $\delta = (2\pi)^{-1-q}$, and we use the notation (5.1b). Let

$$t_j = (j-1)\pi/n, \quad \lambda_j = f(t_j) \qquad (1 \leqslant j \leqslant n).$$

The theorem will be proved with

$$B = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n.$$

(2) For $\zeta \in \mathbb{C} \setminus W$ the matrix $\zeta I - B$ is invertible, and using the notation (5.2a) we obtain

$$\begin{aligned}(\zeta I - B)^{-1} &= \sum_{j=1}^{n} \phi_\zeta(t_j) P_j \\ &= \sum_{j=1}^{n-1} \left([\phi_\zeta(t_j) - \phi_\zeta(t_{j+1})] \sum_{k=1}^{j} P_k \right) + \phi_\zeta(t_n) \sum_{k=1}^{n} P_k.\end{aligned}$$

Since

$$\|P_1 + \cdots + P_j\| = 1 \qquad \text{for} \quad 1 \leqslant j \leqslant n,$$

there follows

$$\|(\zeta I - B)^{-1}\| \leqslant |\phi_\zeta(t_n)| + \sum_{j=1}^{n-1} |\phi_\zeta(t_j) - \phi_\zeta(t_{j+1})| \leqslant \mathrm{d}(\zeta, W)^{-1} + \Lambda_\zeta,$$

where Λ_ξ is given by (5.2b). Let K be as in Lemma 5.2. We see that B satisfies (1.1) with $L = 1 + K$. This constant L depends only on q (and not on n).

(3) One easily sees that

$$\|B^n\| = \left\| \sum_{j=1}^{n} (\lambda_j)^n P_j \right\| = |(\lambda_n)^n| + \sum_{j=1}^{n-1} \left|(\lambda_j)^n - (\lambda_{j+1})^n\right|.$$

Putting $g(t) = (1 - \delta t^{1+q})^n$, we thus obtain

$$\|B^n\| = g(t_n) + \sum_{j=1}^{n-1} \left[g(t_j) + g(t_{j+1})\right] \geqslant \sum_{j=1}^{n} g(t_j) \geqslant \frac{n}{\pi}\int_0^\pi g(t)\, dt.$$

Since $1 - x \geqslant \exp(-2x)$ for $0 \leqslant x \leqslant \frac{1}{2}$, we have

$$g(t) \geqslant \exp(-2\delta n t^{1+q}) \qquad \text{for} \quad 0 \leqslant t \leqslant \pi,$$

and therefore

$$\|B^n\| \geqslant c n^p \qquad \text{with} \quad p = 1 - \frac{1}{1+q}, \quad c = \frac{1}{\pi}\int_0^\pi \exp(-2\delta x^{1+q})\, dx.$$

Since c depends only on q, the proof of the theorem is complete. ■

4. STABILITY ESTIMATES UNDER THE HILLE-YOSIDA RESOLVENT CONDITION

The following theorem gives stability estimates under the condition (1.2). The proof relies on Theorem 2.6 and an integral representation for B^n that has similarity to one used by Bonsall and Duncan (1980, p. 41).

THEOREM 4.1. *Let ξ_j and q be as in Section 2.1, and W of type q. Then there is a constant γ, depending only on W and q, such that*

$$\|B^n\| \leqslant \gamma L n^p \quad (\textit{for } n \geqslant 1), \qquad \textit{with} \quad p = \max\left\{0, \frac{1}{2} - \frac{1}{1+q}\right\},$$

whenever $L > 0$, $s \geqslant 1$, $\|\cdot\|$ *is a norm on the vector space* $\mathbb{C}^{s,s}$, *and the* $s \times s$ *matrix* B *satisfies* (1.2).

Proof. We assume, with no loss of generality, that $q \geqslant 1$. Let B satisfy (1.2).

For any $\rho > 0$ we have the integral representation

$$B^n = \frac{n!}{2\pi i} \int_\Gamma (\bar{\zeta})^{-n-1} \exp(\bar{\zeta} B)\, d\bar{\zeta}, \tag{4.1}$$

where Γ is the positively oriented circle $|\zeta| = \rho$, and $\bar{\zeta}$ denotes the complex conjugate of ζ.

By virtue of Theorem 2.6 there is a set $V = \operatorname{conv}(W_\delta)$, depending only on W and q, such that

$$\|\exp(\bar{\zeta} B)\| \leqslant L \exp[\operatorname{Re}(\xi \bar{\zeta})] \qquad \text{for} \quad \zeta = \rho e^{i\omega} \tag{4.2}$$

if ω is a normal direction to V at $\xi \in \partial V$.

We shall use the notation of Section 2.1, and we put

$$\Gamma_j = \Gamma \cap S_j \qquad \text{for} \quad 1 \leqslant j \leqslant m.$$

Further, we define

$$r(t) = 1 - \delta |t|^{1+q} \qquad \text{for} \quad -\pi \leqslant t \leqslant \pi.$$

Let $\zeta = \rho e^{i\omega} \in \Gamma_j$ be given. In order to apply (4.2) to this ζ we write

$$\zeta = \rho e^{i\theta} \xi_j, \qquad \text{with} \quad -\sigma_{j-1} \leqslant \theta \leqslant \sigma_j.$$

It can be seen that there exists a ξ on $\partial V \cap S_j$ such that ω is a normal direction to V at point ξ. A little calculation shows that ξ can be written in the form

$$\xi = r(t) e^{it} \xi_j, \tag{4.3a}$$

where $t \in (-\sigma_{j-1}, \sigma_j)$ depends continuously on $\theta \in [-\sigma_{j-1}, \sigma_j]$, and

$$t + \alpha(t) = \theta, \qquad |\alpha(t)| < \frac{\pi}{2}, \qquad \tan \alpha(t) = -\frac{r'(t)}{r(t)}. \tag{4.3b}$$

Inserting (4.3a) into (4.2), we obtain

$$\|\exp(\bar{\zeta} B)\| \leqslant L \exp\left[r(t) \rho \operatorname{Re} e^{i(t-\theta)}\right].$$

Applying this inequality for all $\zeta \in \Gamma_j$, there follows

$$\left\| \int_{\Gamma_j} (\bar{\zeta})^{-n-1} \exp(\bar{\zeta} B) \, d\bar{\zeta} \right\| \leqslant L \rho^{-n} \int_{-\sigma_{j-1}}^{\sigma_j} \exp[r(t) \rho] \, d\theta, \qquad (4.4)$$

where, in the last integral, t depends on the integration variable θ via the relations (4.3b).

By differentiating both members of the last equality in (4.3b) one obtains

$$\alpha'(t) = \frac{-r''(t) r(t) + [r'(t)]^2}{[r(t)]^2 + [r'(t)]^2},$$

and, by using the first equality in (4.3b), there follows

$$\frac{d}{dt} \theta = \frac{[r(t)]^2 + 2[r'(t)]^2 - r''(t) r(t)}{[r(t)]^2 + [r'(t)]^2}.$$

The last expression is positive, and can be bounded from above, uniformly for all $t \in [-\pi, \pi]$, by some constant K which depends only on δ and q. Therefore

$$\int_{-\sigma_{j-1}}^{\sigma_j} \exp[r(t) \rho] \, d\theta \leqslant K \int_{-\pi}^{\pi} \exp[r(t) \rho] \, dt = 2 K e^{\rho} \int_0^{\pi} \exp(-\delta \rho t^{1+q}) \, dt$$

$$\leqslant 2 K M e^{\rho} \rho^{-1/(1+q)},$$

with $M = \int_0^\infty \exp(-\delta x^{1+q}) \, dx$.

In view of (4.1), (4.4) we obtain

$$\|B^n\| \leqslant \frac{mKM}{\pi} \cdot L \cdot n! \, \rho^{-n} e^{\rho} \rho^{-1/(1+q)}.$$

Choose $\rho = n$, and recall from Stirling's formula that

$$n! \leqslant (n/e)^n e \sqrt{n}.$$

There follows $\|B^n\| \leqslant \gamma L n^p$, $p = \frac{1}{2} - 1/(1+q)$, with γ depending only on W and q. ■

5. TECHNICAL LEMMAS

In the following we assume that ξ_j and q are as in Section 2.1, and we use the notation of that section.

Our first lemma provides us with a transparent upper bound for the quantity $\mathrm{d}(\zeta, W)^{-1}$ occurring in (1.1) when $W = W_\delta$. This upper bound has been used in the proofs of Theorem 2.4 and Theorem 3.1.

LEMMA 5.1. *Let W_δ be given by* (2.3), *and $\zeta \in S_j \setminus W_\delta$, $\theta = \operatorname{Arg}(\zeta \xi_j^{-1})$. Then*

$$\mathrm{d}(\zeta, W_\delta) \geqslant c\big(|\zeta| - 1 + \delta|\theta|^{1+q}\big) \qquad \textit{with} \quad c = \frac{1 - \delta\pi^{1+q}}{1 + \delta(1+q)\pi^q}.$$

Proof. We first deal with the case where $m = 1$ and $\xi_1 = 1$. Since W_δ is symmetric with respect to the real axis, it is sufficient to consider $\zeta = |\zeta|e^{i\theta}$ with $0 \leqslant \theta \leqslant \pi$, $|\zeta| > 1 - \delta\theta^{1+q}$.

With the definitions

$$r(t) = 1 - \delta t^{1+q}, \qquad f(t) = r(t)e^{it}, \quad \text{and} \quad V = \{f(t) : 0 \leqslant t \leqslant \pi\},$$

we have

$$\mathrm{d}(\zeta, W_\delta) = \mathrm{d}(\zeta, V).$$

In order to estimate $\mathrm{d}(\zeta, V)$ we consider the tangent line to V at point $f(t)$, given by

$$l(t) = \{\xi : \xi = f(t) + \lambda f'(t) \text{ with } -\infty < \lambda < \infty\}.$$

Since $f'(t) = [r'(t) + ir(t)]e^{it}$, we see that the angle $\beta(t)$ between the line $l(t)$ and the positive real axis can be written in the form

$$\beta(t) = \gamma(t) + t \qquad \text{with} \quad \gamma(t) = \operatorname{Arg}[r'(t) + ir(t)].$$

For $0 \leqslant t \leqslant \pi$ we have $r(t) > 0$ and $r'(t) \leqslant 0$, so that

$$\frac{\pi}{2} \leqslant \gamma(t) < \pi \qquad \text{with} \quad \tan\gamma(t) = \frac{r(t)}{r'(t)}.$$

By differentiating both members of the last equality one obtains

$$\gamma'(t) = \frac{r'(t)r'(t) - r''(t)r(t)}{r'(t)r'(t) + r(t)r(t)},$$

so that $\gamma'(t) > 0$ for $0 < t \leqslant \pi$. Consequently, $\beta(t)$ is monotonic increasing for $0 \leqslant t \leqslant \pi$.

This monotonicity can be seen to imply

$$\mathrm{d}(\zeta, V) \geqslant |\zeta - \zeta_0|,$$

where ζ_0 is the projection of $\zeta = |\zeta| e^{i\theta}$ on the line $l(\theta)$. By using the formulas

$$|\zeta - \zeta_0| = |\zeta - f(\theta)| \sin[\pi - \gamma(\theta)],$$

$$\sin \gamma(\theta) = r(\theta)\{r(\theta)^2 + [r'(\theta)]^2\}^{-1/2}$$

one easily arrives at the inequality in Lemma 5.1.

It is evident that the inequality established for $m = 1$, $\xi_1 = 1$ is also valid for $m = 1$ and general ξ_1 with $|\xi_1| = 1$.

By using some simple geometric arguments one sees that the inequality proved for $m = 1$ is still valid for general $m \geqslant 1$. ■

Our second lemma deals with $W = W_\delta$, where $m = 1$, $\xi_1 = 1$, i.e.

$$W = \{\xi : \xi \in \mathbb{C} \text{ with } |\xi| \leqslant 1 - \delta |\operatorname{Arg}(\xi)|^{1+q}\},$$
$$q \geqslant 0, \quad 0 < \delta < \pi^{-1-q}. \tag{5.1a}$$

Clearly, the boundary of W is equal to the range of the function

$$f(t) = (1 - \delta |t|^{1+q}) e^{it} \qquad \text{for} \quad -\pi \leqslant t \leqslant \pi. \tag{5.1b}$$

For any given $\zeta \in \mathbb{C} \setminus W$, the function

$$\phi_\zeta(t) = [\zeta - f(t)]^{-1} \qquad \text{for} \quad -\pi \leqslant t \leqslant \pi \tag{5.2a}$$

specifies a curve with length

$$\Lambda_\zeta = \int_{-\pi}^{\pi} |\phi'_\zeta(t)| \, \mathrm{d}t. \tag{5.2b}$$

Since

$$\Lambda_\zeta = \int_{-\pi}^{\pi} |f'(t)| \cdot |\zeta - f(t)|^{-2} \, \mathrm{d}t \tag{5.3}$$

and $|\zeta - f(t)|^{-2} \leqslant \mathrm{d}(\zeta, W)^{-2}$, we see that $\Lambda_\zeta = \mathscr{O}(\mathrm{d}(\zeta, W)^{-2})$.

The following lemma provides a distinct improvement over the last estimate, for ζ close to W. The lemma has been essential in the proof of Theorem 3.3.

LEMMA 5.2. *Assume* (5.1), (5.2). *Then there is a constant K, depending only on δ and q, such that for all $\zeta \in \mathbb{C} \setminus W$*

$$\Lambda_\zeta \leqslant K \cdot \mathrm{d}(\zeta, W)^{-1}. \tag{5.4}$$

Proof.

(1) From (5.3) we have $\Lambda_\zeta \leqslant 2\pi\mu_1 \mathrm{d}(\zeta, W)^{-2}$, with

$$\mu_1 = \max\{|f'(t)| : -\pi \leqslant t \leqslant \pi\}.$$

Therefore, it is sufficient to prove (5.4) for ζ close to W. Moreover, by using a compactness argument, one sees that it is sufficient to prove (5.4) for ζ in some neighborhood of each individual $\xi \in \partial W$. Let $\xi_0 \in \partial W$ be given. We have to show that there is an $\epsilon > 0$ such that (5.4) holds for all ζ belonging to

$$V(\epsilon) = \{\zeta : \zeta \in \mathbb{C} \setminus W \ \& \ |\zeta - \xi_0| < \epsilon\}.$$

(2) We first assume $\xi_0 = f(t_0)$ with $-\pi < t_0 < \pi$. For any $\tau > 0$ we have

$$\Lambda_\zeta = I_0 + I_1$$

with

$$I_0 = \int_{\mathrm{T}_0} |f'(t)| \cdot |\zeta - f(t)|^{-2} \, \mathrm{d}t, \qquad I_1 = \int_{\mathrm{T}_1} |f'(t)| \cdot |\zeta - f(t)|^{-2} \, \mathrm{d}t,$$

where

$$\mathrm{T}_0 = \{t : -\pi \leqslant t \leqslant \pi, |t - t_0| \geqslant \tau\},$$

$$\mathrm{T}_1 = \{t : -\pi \leqslant t \leqslant \pi, |t - t_0| \leqslant \tau\}.$$

We choose $\tau > 0$ so small that

$$|f'(t) - f'(\tilde{t})| \leqslant \tfrac{1}{2}|f'(t)|$$
$$\text{for all} \quad t, \tilde{t} \in [-\pi, \pi] \quad \text{with } |t - \tilde{t}| \leqslant 2\tau. \tag{5.5}$$

Since $f(t) \neq \xi_0$ on T_0, we can choose $\epsilon > 0$ with

$$|f(t) - \xi_0| \geqslant 2\epsilon \qquad \text{for all} \quad t \in T_0. \tag{5.6}$$

Let $\zeta \in V(\epsilon)$ be given. In view of (5.6) we have, for all $t \in T_0$,

$$|f(t) - \zeta| \geqslant |f(t) - \xi_0| - |\zeta - \xi_0| \geq \epsilon.$$

Hence, the integrand of I_0 can be bounded by $\mu_1 \epsilon^{-2}$, which yields

$$I_0 \leqslant K_0 \mathrm{d}(\zeta, W)^{-1} \qquad \text{with} \quad K_0 = 2\pi\mu_1\epsilon^{-1}. \tag{5.7}$$

In order to bound I_1 we choose t_ζ such that

$$t_\zeta \in T_1 \quad \text{and} \quad |\zeta - f(t_\zeta)| = \min\{|\zeta - f(t)| : t \in T_1\}.$$

For $t \in T_1$ we have

$$|\zeta - f(t)| = |\zeta - f(t_\zeta)| \cdot \left| 1 - \frac{f(t) - f(t_\zeta)}{\zeta - f(t_\zeta)} \right|$$
$$= |\zeta - f(t_\zeta)| \cdot \left| 1 - \frac{f'(t_\zeta)(t - t_\zeta)}{\zeta - f(t_\zeta)} [1 + \alpha(t)] \right|,$$

where, in view of (5.5),

$$|\alpha(t)| \leqslant \tfrac{1}{2}.$$

Using this inequality and the fact that

$$\left| \operatorname{Arg}\left(\frac{f'(t_\zeta)(t - t_\zeta)}{\zeta - f(t_\zeta)} \right) \right| \geqslant \frac{\pi}{2},$$

one obtains, after a small calculation,

$$|\zeta - f(t)|^{-2} \leqslant 4|\zeta - f(t_\zeta)|^{-2}(1 + x^2)^{-1},$$

where

$$x = |\zeta - f(t_\zeta)|^{-1}|f'(t_\zeta)|(t - t_\zeta).$$

Inserting the last inequality into the integrand of I_1 there follows

$$I_1 \leqslant 4\mu_1|f'(t_\zeta)|^{-1}|\zeta - f(t_\zeta)|^{-1}$$
$$\times \int_{T_1}(1 + x^2)^{-1}\,\mathrm{d}\Big(|\zeta - f(t_\zeta)|^{-1}|f'(t_\zeta)|(t - t_\zeta)\Big).$$

From this we obtain

$$I_1 \leqslant K_1 \mathrm{d}(\zeta, W)^{-1}, \tag{5.8}$$

with

$$K_1 = 4\mu_0^{-1}\mu_1\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{1 + x^2}, \qquad \mu_0 = \min\{|f'(t)| : -\pi \leqslant t \leqslant \pi\}.$$

Combining (5.7) and (5.8), we obtain (5.4), with $K = K_0 + K_1$, for all $\zeta \in V(\epsilon)$.

(3) We finally assume $\xi_0 = f(-\pi) = f(\pi)$. We choose $\tau > 0$ according to (5.5), and we write

$$\Lambda_\zeta = I_0 + I_1 + I_2$$

$$\text{with} \quad I_j = \int_{T_j}|f'(t)|\cdot|\zeta - f(t)|^{-2}\,\mathrm{d}t \quad (j = 0, 1, 2).$$

Here

$$T_0 = \{t : |t| \leqslant \pi - \tau\}, \qquad T_1 = \{t : -\pi \leqslant t \leqslant -\pi + \tau\},$$
$$T_2 = \{t : \pi - \tau \leqslant t \leqslant \pi\}.$$

We choose $\epsilon > 0$ again according to (5.6), which yields (5.7). Both I_1 and I_2 can be bounded in the same way as I_1 was bounded in part (2). We arrive at $I_1 \leqslant K_1 d(\zeta, W)^{-1}$, $I_2 \leqslant K_1 d(\zeta, W)^{-1}$, with K_1 as defined in (5.8). Hence, (5.4) holds, with $K = K_0 + 2K_1$, for all $\zeta \in V(\epsilon)$. ∎

REFERENCES

Bonsall, F. F. and Duncan, J. 1980. Numerical ranges, in *Studies in Functional Analysis*, (R. G. Bartle, Ed.), Math. Assoc. Amer., pp. 1–49.

Dorsselaer, J. L. M. van, Kraaijevanger, J. F. B. M., and Spijker, M. N. 1993. Linear stability analysis in the numerical solution of initial value problems, *Acta Numer.* 1993:199–237.

Dowson, H. R. 1978. *Spectral Theory of Linear Operators*, Academic, London.

Friedland, S. 1981. A generalization of the Kreiss matrix theorem, *SIAM J. Math. Anal.* 12:826–832.

Gorelick, M. R. and Kranzer, H. C. 1976. An extension of the Kreiss stability theorem to families of matrices of unbounded order, *Linear Algebra Appl.* 14:237–255.

Kraaijevanger, J. F. B. M. 1994. Two counterexamples related to the Kreiss matrix theorem, *BIT* 34:113–119.

Lenferink, H. W. J. and Spijker, M. N. 1990. A generalization of the numerical range of a matrix, *Linear Algebra Appl.* 140:251–266.

Lenferink, H. W. J. and Spijker, M. N. 1991. On the use of stability regions in the numerical analysis of initial value problems, *Math. Comp.* 57:211–220.

Lubich, C. and Nevanlinna, O. 1991. On resolvent conditions and stability estimates, *BIT* 31:293–313.

McCarthy, C. A. 1993. A strong resolvent condition need not imply power-boundedness, in preparation.

Palencia, C. and García-Archilla, B. 1993. Stability of linear multistep methods for sectorial operators in Banach spaces, *Appl. Numer. Math.* 12:503–520.

Pazy, A. 1983. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York.

Reddy, S. C. and Trefethen, L. N. 1990. Lax-stability of fully discrete spectral methods via stability regions and pseudo-eigenvalues, *Comput. Methods Appl. Mech. Engrg.* 80:147–164.

Reddy, S. C. and Trefethen, L. N. 1992. Stability of the method of lines, *Numer. Math.* 62:235–267.

Richtmyer, R. D. and Morton, K. W. 1967. *Difference Methods for Initial Value Problems*, 2nd ed., Wiley, New York.

Spijker, M. N. 1991. On a conjecture by LeVeque and Trefethen related to the Kreiss matrix theorem, *BIT* 31:551–555.

Spijker, M. N. 1993. Numerical ranges and stability estimates, *Appl. Numer. Math.* 13:241–249.

Spijker, M. N. and Straetemans, F. A. J. 1995. Linear stability analysis and stability regions in the numerical solution of initial value problems, submitted for publication.

Tadmor, E. 1981. The equivalence of L_2-stability, the resolvent condition, and strict H-stability, *Linear Algebra Appl.* 41:151–159.

Tadmor, E. 1986. The resolvent condition and uniform power-boundedness, *Linear Algebra Appl.* 80:250–252.

Trefethen, L. N. 1995. *Spectra and Pseudospectra*, in preparation.

Wegert, E. and Trefethen, L. N. 1994. From the Buffon needle problem to the Kreiss matrix theorem, *Amer. Math. Monthly* 101:132–139.

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