MA3D5 Galois Theory assignment sheet 1

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The deadline for these problems is **5:00 pm on Thursday 20 September**. Please hand in your work to the MA3D5 Galois Theory box outside the Undergraduate Office. Prove everything you claim, and give references and names of theorems whenever appropriate.

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1. (a) Is $\mathbf{Z}/27\mathbf{Z}$ a field? (Prove your answer.)

   (b) Give two distinct polynomials $f_1, f_2 \in \mathbf{F}_3[X]$ such that $f_i(x) = 1$ for all $x \in \mathbf{F}_3$.

   (c) Give an irreducible polynomial of degree 3 in $\mathbf{F}_3[X]$. (Don't forget to prove that it is irreducible.)

   (d) Give an example of a field with 27 elements. What is its characteristic? What is its prime subfield?

2. (a) Prove that $X^9 + 6X^2 + 18 \in \mathbf{Z}[X]$ is Eisenstein. Deduce that it is irreducible in $\mathbf{Z}[X]$ and $\mathbf{Q}[X]$.

   (b) Prove that $X^3 + X + 1 \in \mathbf{F}_7[X]$ is irreducible. Conclude that $X^3 - 6X + 8 \in \mathbf{Q}[X]$ is irreducible.

   (c) Prove that $g(X) = X^4 + 1 \in \mathbf{Q}[X]$ is irreducible. Hint: what can you say about $g(X + 1)$?

3. (a) Find the minimal polynomials over $\mathbf{Q}$ of the complex numbers i, $\sqrt{2}$, $i\sqrt{2}$ and $\alpha = \frac{1+i}{\sqrt{2}}$.

   (b) What are the degrees of these algebraic numbers?

   (c) List the conjugates in $\mathbf{C}$ of these algebraic numbers.

   (d) Prove $\mathbf{Q}(\sqrt{2}, i) = \mathbf{Q}(\alpha)$. Hint: what can you say about α^2 and $(\alpha^2 - 1)\alpha$?

4. (a) Let α denote the image of X in $\mathbf{Q}[X]/(X^3 + 2X + 2)$. Express each of α^3, $1/\alpha$, and $1/(\alpha^2 + 1)$ in the form $a + b\alpha + c\alpha^2$ with $a, b, c \in \mathbf{Q}$.

   (b) Let β denote the image of X in $\mathbf{F}_2[X]/(X^3 + X + 1)$. Express each of β^4 and $1/(\beta + 1)$ in the form $a + b\beta + c\beta^2$ with $a, b, c \in \{0, 1\}$.