MA3D5 Galois Theory assignment sheet 2

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The deadline for these problems is **4:00 pm on Thursday 3 November**. Please hand in your work to the MA3D5 Galois Theory box outside the Undergraduate Office. Prove everything you claim, and give references and names of theorems whenever appropriate.

1. Determine the number of ring homomorphisms (don't forget to prove your answers)
   - (a) $\mathbf{F}_2 \to \mathbf{F}_3$
   - (b) $\mathbf{Q}[X]/(X^7 - 3) \to \mathbf{Q}[X]/(X^8 + 4X^5 - 6X + 2)$
   - (c) $\mathbf{F}_7[X]/(X^2 + X - 1) \to \mathbf{F}_7[X]/(X^2 + 1)$
   - (d) $\mathbf{Q}(\sqrt[4]{2}) \to \mathbf{C}$
2. Let $K = \mathbf{Q}(\sqrt[4]{2})$.
   - (a) Which of the morphisms from 1(d) are $\mathbf{Q}(\sqrt{2})$-homomorphisms?
   - (b) And which are K-homomorphisms?
   - (c) Determine the automorphism group $\mathrm{Aut}(K/\mathbf{Q})$.
   - (d) Find an element in K that is not in $\mathbf{Q}$ and that is fixed by every element of $\mathrm{Aut}(K/\mathbf{Q})$.
   - (e) Conclude that $K/\mathbf{Q}$ is not Galois.
3. Let $\zeta = e^{2\pi i/7}$ be a *primitive 7th root of unity* in $\mathbf{C}$. You may use without proof that its minimal polynomial is $X^6 + X^5 + \cdots + X + 1$.
   - (a) Express $x = \mathrm{Re}(\zeta) = \cos(2\pi/7)$ as a linear combination of ζ and ζ^{-1}, and manipulate to get a non-zero quadratic polynomial with coefficients in $\mathbf{Q}(x)$ of which ζ is a root. Deduce $[\mathbf{Q}(\zeta) : \mathbf{Q}(x)] \in \{1, 2\}$. What does this tell us about $[\mathbf{Q}(x) : \mathbf{Q}]$?
   - (b) Prove that it is not possible to construct a regular heptagon (7-sided polygon) with ruler and compass starting from two points in the plane. [You may use without proof that there is no loss of generality in assuming that the two starting points are $(0, 0)$ and $(1, 0)$, that the heptagon's centre is $(0, 0)$, and that $(1, 0)$ is one of its vertices.]
4. Let K be a field of characteristic different from 2 and let L/K be a quadratic extension (i.e., an extension of degree 2). We know from exercise 20 of the non-assessed problems that L/K is of the form $L = K(\alpha)$ with $\alpha \in L$ a root of some polynomial $aX^2 + bX + c$.
   - (a) Prove that the discriminant $D = b^2 - 4ac \in K$ has a square root $\sqrt{D}$ in L and that we have $L = K(\sqrt{D})$. [Hint: use the quadratic formula.]
   - (b) Prove that L/K has two distinct automorphisms.
   - (c) Conclude that L/K is Galois.