MA3D5 Galois Theory assignment sheet 3

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The deadline for these problems is **4:00 pm on Thursday 17 November**. Please hand in your work to the MA3D5 Galois Theory box outside the Undergraduate Office. Prove everything you claim, and give references and names of theorems whenever appropriate.

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1. Let ω be a primitive 3rd root of unity in $\mathbf{C}$, let $\alpha = \sqrt[3]{7}$, and let $K = \mathbf{Q}(\omega, \alpha)$.
   (a) Prove that K has degree 6 (over $\mathbf{Q}$).
   (b) Prove that K has an automorphism σ such that $\sigma(\alpha) = \alpha$ and $\sigma(\omega) = \omega^2$ and an automorphism τ such that $\tau(\omega) = \omega$ and $\tau(\alpha) = \omega\alpha$.
   (c) Prove that σ has order 2, that τ has order 3, and that we have $\sigma\tau\sigma = \tau^2$.
   (d) Prove that K is Galois over $\mathbf{Q}$ with Galois group isomorphic to S_3.
2. Continue in the situation of 2.
   (e) Find all subgroups of S_3, draw the diagram of inclusions and indicate the orders of the subgroups. Which subgroups are normal?
   (f) Draw the corresponding diagram of subfields of K, say which subfield corresponds to which of the groups from (e), and give the degrees of the fields over $\mathbf{Q}$.
   (g) Which of the fields M from (f) are Galois over $\mathbf{Q}$? For which of the fields M from (f) is K/M Galois?
3. Suppose that the polynomial $f = aX^4 + bX^2 + c$ for some $a, b, c \in \mathbf{Q}$ is irreducible over $\mathbf{Q}$. Added correction: assume further that a is non-zero.

   Let S be a splitting field over $\mathbf{Q}$ of f. Prove that $[S : \mathbf{Q}]$ is either 4 or 8.

Please note that the fourth problem is on the other side!

Recall that the lecture on Friday 28 October claimed the following result, and that this result was used in the proof of the fundamental theorem of Galois theory on Tuesday 1 November.

Proposition 78. *Let L/K be a field extension and $H \subset G \subset \mathrm{Aut}(L/K)$ subgroups. If $[G : H]$ is finite, then $[L^H : L^G] \leq [G : H]$.*

In problem 4, you will prove this result in the case where the extension L/K is finite, which is all that counts in this module. You may use all results that we proved in class up to and including Artin's lemma.

Artin's Lemma. *Proposition 78 is true in the case $H = 1$. In other words, if L/K is a field extension and $G \subset \mathrm{Aut}(L/K)$ is a finite subgroup, then we have $[L : L^G] \leq \#G$.*

4. (a) Prove that equality holds in Artin's lemma, i.e., that we also have $[L : L^G] \geq \#G$.
   (b) Use (a) and the tower law to prove Proposition 78 in the case where G is finite. Note that in this case, we have equality, i.e., $[L^H : L^G] = [G : H]$.
   (c) Conclude from (b) that Proposition 78 is true if L/K is a finite extension
   (d) (Optional, does not count for your mark!) Prove that Proposition 78 is true as stated, without requiring G or the extension L/K to be finite. Hint: recall and/or prove that G has a normal subgroup J of finite index contained in H, and replace L by L^J. What happens to G and H?