MA3D5 Galois Theory assignment sheet 4

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The deadline for these problems is **4:00 pm on Thursday 1 December**. Please hand in your work to the MA3D5 Galois Theory box outside the Undergraduate Office. Prove everything you claim, and give references and names of theorems whenever appropriate.

1. Find all subfields of $K = \mathbf{Q}(\sqrt[6]{2})$, where $\sqrt[6]{2}$ is the positive sixth root of 2 in $\mathbf{R}$. [Hint: $K/\mathbf{Q}$ is not Galois, so find a good field M to do Galois theory with. To save yourself some work, don't list all subgroups of $\mathrm{Gal}(M/\mathbf{Q})$ but only the ones that correspond to fields contained in K.]

2. In this exercise we shall see an example of a finite extension of fields that is not primitive. Let p be a prime number and $K = \mathbf{F}_p(X, Y) = (\mathbf{F}_p(X))(Y)$ the field of rational functions in two variables over $\mathbf{F}_p$.

   (a) Let $f \in K[T]$ be the polynomial $T^p - X$. Let L/K be a splitting field for f. How many roots does f have in L, and what are their multiplicities? Show that f is irreducible over K and inseparable, and deduce that L/K is an inseparable extension of degree p.

   (b) Now let $g \in L[U]$ be the polynomial $U^p - Y$. Let M be a splitting field of g over L. Show that M/L is inseparable of degree p, and deduce $[M : K] = p^2$.

   (c) Show that any $\alpha \in M$ satisfies $\alpha^p \in K$. Deduce that M/K is not primitive.

3. Let $f = X^3 + X^2 + 1$ and $g = Y^3 + Y + 1$ be polynomials over $\mathbf{F}_5$.

   (a) Show that f and g are both irreducible over $\mathbf{F}_5$. Show that the fields $K = \mathbf{F}_5[X]/(f)$ and $L = \mathbf{F}_5[Y]/(g)$ are isomorphic.

   (b) Let $\alpha \in K$ be the class of X and $\beta \in L$ the class of Y, so that $f(\alpha) = 0$ and $g(\beta) = 0$. Let $\gamma = 2\alpha^2 + \alpha + 3 \in K$. What is $g(\gamma)$? Hint: to check if your answer is correct, see if (c) makes sense.

   (c) Deduce that there is a field isomorphism $\phi : L \to K$ with $\phi(\beta) = \gamma$.

   (d) Give the set of all $\psi(\beta) \in K$ as ψ ranges over the set of field isomorphisms $L \to K$. Hint: what are the automorphisms of K? You don't need to express $\psi(\beta)$ as a linear combination of $\alpha^2, \alpha, 1$, just give a formula from which it can be computed.

4. (a) Give a generator of $\mathbf{F}_{13}^* = (\mathbf{Z}/13\mathbf{Z})^*$.

   (b) Give the unique subgroup of order 3 of $(\mathbf{Z}/13\mathbf{Z})^*$.

   Consider the regular tridecagon T (13-sided polygon) in the plane with centre $O = (0, 0)$ and a vertex $V = (1, 0)$. We know (exactly as in 3(b) of assignment sheet 2) that it is not possible to construct T by ruler and compass starting from only $\{O, V\}$. Label the vertices $V_0, V_1, \ldots, V_{12}$ starting with $V_0 = V$ and going anti-clockwise.

   Let $C = \frac{1}{3}(V_1 + V_3 + V_9)$ be the centre of mass of the triple of vertices $\{V_1, V_3, V_9\}$.

   (c) Prove that it *is* possible to construct C starting from only $\{O, V\}$. Hint: how does the number $\zeta_{13} + \zeta_{13}^3 + \zeta_{13}^9$ relate to C, and why is (b) part of this problem?