BONUS EXAMPLE SHEET 1

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Abstract. Extra exercises for fun and profit; optional and not assessed. All except for (4) are somewhat content-free. Throughout this example sheet, R will denote a ring; as in the lectures, all rings are assumed commutative unital, and all ring homomorphisms are unital (send the unit to the unit). If there are any definitions that you don't know, Wikipedia is your friend. Any questions email d.s.t.holmes@warwick.ac.uk.

1) Show that there exists a unique homomorphism from $\mathbb{Z}$ to R. We shall call it $\phi : \mathbb{Z} \to R$ for the remainder of the sheet.

2) Show that if R is a field then the characteristic of R is equal to a non-negative generator of the kernel of ϕ. (This allows us to generalise the definion of the characteristic to any ring in an obvious way).

Definition 1. *We say that R is an* <u>*integral domain*</u> *(or just a domain, or just integral) if for any a, $b \in R$ such that $ab = 0$, we have that either $a = 0$ or $b = 0$.*

3) Show that every field is an integral domain.

4) Suppose R is an integral domain. Show that there exists a field $F(R)$ with an injective ring homomorphism $\psi : R \to F(R)$ such that for any injective ring homomorphism f from R to a field K, we can factor f via ψ uniquely; in other words, such that there exists a unique ring homomorphism $g : F(R) \to K$ such that the following commutes:

$$\begin{array}{ccc} R & \xrightarrow{\psi} & F(R) \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle g} \\ & & K \end{array}$$

- in other other words, such that $f = g \circ \psi$.

We call $F(R)$ the <u>field of fractions</u> of R. The next exercise shows that it is essentially well defined. [Hint: construct elements of $F(R)$ as 'fractions' of elements of R. The key is to give the right definition of equality of two such fractions. If you get stuck, look at 'Field of fractions' on Wikipedia.]

5) Show that if $F'(R)$ is another field satisfying the property above, then there is a unique isomorphism from $F(R)$ to $F'(R)$ which is compatible with the maps from R. [Hint: this is an entirely formal consequence of the previous question. The above definition is known as a 'universal property', and the fact that such a property yields an essentially unique solution is something that is used all the time].

6)[Sanity check] Show that the field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$. Show that the field of fractions of a field K is again K.

7) Show that if R is a field of non-zero characteristic then the image of ϕ is equal to the prime subfield of R. Given an example to show that this can fail in characteristic zero (in fact it always will).

8) Give an example of:
a) a ring that is not an integral domain.
b) an integral domain that is not a field.
In each case, can you give an example with characteristic zero and an example with characteristic non-zero (where we define the characteristic to be the non-negative generator of the kernel of ϕ)?

Note that there does exist a definition of a 'field of fractions' for any ring, not necessarily integral. If you are interested, look up 'total quotient ring' on Wikipedia. However, I have ommitted it as it is not really the right thing to study in my experience.