MA3D5 HANDOUT: RATIONAL FUNCTIONS

Executive summary. Let K be a field. The field $K(X)$ (note the round brackets, not square) of *rational functions* in X with coefficients in K consists of fractions f/g, where $f, g \in K[X]$ and $g \neq 0$.

The whole story. To every integral domain[1] we can associate a field, called the *field of fractions*. For example, the field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$. Let us recall some elementary properties of $\mathbb{Q}$:

- $\mathbb{Q}$ is a field.
- $\mathbb{Q}$ consists of fractions a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$.
- Not all of these symbols refer to distinct elements of $\mathbb{Q}$: for example, $2/3$ and $(-4)/(-6)$ both refer to the same element of $\mathbb{Q}$.
- There is an embedding (i.e. an injective ring homomorphism) $\mathbb{Z} \to \mathbb{Q}$ given by $a \mapsto a/1$.

The construction which builds $\mathbb{Q}$ out of $\mathbb{Z}$ works with any integral domain, and is how we define the field of fractions.

So let R be an integral domain. We define a set S to consist of "fractions" with numerator and denominator in R:

$$S = \{(a, b) \mid a, b \in R, b \neq 0\}.$$

We will think (and usually write) a/b instead of (a, b). Now we define an equivalence relation on S, which says which fractions will define the same element of the fraction field.

$$(a, b) \sim (c, d) \text{ if and only if } ad = bc.$$

Define the field of fractions of R, written $\mathrm{Frac}(R)$, to be the set of equivalence classes of fractions: $\mathrm{Frac}(R) := S/\sim$.

At the moment $\mathrm{Frac}(R)$ is only a set; we need to define operations of addition and multiplication. Following the example of rational numbers, we do this as follows:

$$(a, b) + (c, d) = (ad + bc, bd) \text{ and } (a, b) \times (c, d) = (ac, bd).$$

It is straightforward to verify that this is well defined on equivalence classes, that is, that, if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then

$$(a, b) + (c, d) \sim (a', b') + (c', d') \text{ and } (a, b) \times (c, d) \sim (a', b') \times (c', d').$$

Now all that remains is to verify the following facts, all of which are easy:

- $\mathrm{Frac}\, R$ is a ring under the operations $+$ and $\times$;
- the elements $(0, b)$ are all equivalent, and their equivalence class is the zero element of $\mathrm{Frac}\, R$;
- every non-zero element has an inverse: explicitly, the inverse of the class of (a, b) is the class of (b, a);
- the map $i\colon R \to \mathrm{Frac}\, R$ defined by $a \mapsto (a, 1)$ is an injective ring homomorphism.

In fact $\mathrm{Frac}\, R$ is the *smallest* field into which R embeds, in the following strong sense: if $f\colon R \to F$ is any injective ring homomorphism from R to a field F, then there is a unique injection of fields $g\colon \mathrm{Frac}\, R \to F$ such that $f = g \circ i$.

Now we can define the field of rational functions over a field K.

Definition. Let K be a field. The *field of rational functions in X over K* (or *with coefficients in K*) is the field $K(X) := \mathrm{Frac}(K[X])$.

[1] Recall that an *integral domain* is a ring R with $0 \neq 1$ and with no zero-divisors, i.e. if $a, b \in R$ and $ab = 0$ then either $a = 0$ or $b = 0$.