Lower bounds on heights of points on elliptic curves

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Lang's height conjecture

Conjecture (Lang)

Let E be an elliptic curve over a number field K and let $P \in E(K)$ be a non-torsion point. Then

$$\widehat{h}(P) \geq C_1 \log(\mathrm{Norm}(\Delta)) - C_2$$

with C_1 and C_2 depending only on K.

Theorem (Hindry-Silverman)

If instead K is a function field of characteristic 0 and genus g, then

$$\widehat{h}(P) \geq \frac{1}{10^{12.6+23g}} \deg(\Delta).$$

Theorem (Naskręcki-Streng)

$O(g^3)$ instead of $O(10^{23g})$

Definitions

Let

- $k = \overline{k}$ field of characteristic 0 (for simplicity),
- C/k smooth, projective, geometrically irreducible curve,
- $K = k(C)$, E/K elliptic curve, $P \in E(K)$.

Heights:

- $P = (x(P), y(P))$, with $x(P) \in k(C)$,
- $h(P) := \frac{1}{2}\deg(x(P))$,
- $\widehat{h}(P) := \lim_{n\to\infty} \frac{1}{n^2} h(nP)$.

The discriminant:

- For v a discrete valuation of K, write $E : y^2 = x^3 + Ax + B$ (v-minimal Weierstrass eqn.), and $\Delta_v := -16(4A^3 + 27B^2)$.
- $\Delta := \sum_v v(\Delta_v)[v] \in \mathrm{Div}(C)$. Assume $\Delta \neq 0$.

Lang's conjecture:
(number field)
$\widehat{h}(P) \geq C_1 \log(N(\Delta)) - C_2$.

Hindry-Silverman:
(function field)
$\widehat{h}(P) \geq \frac{1}{10^{12.6+23g}} \deg(\Delta)$.

Naskręcki-Streng:
(function field)
$O(g^3)$ instead of $O(10^{23g})$.

Definitions (2)

Conductor:

- $N := \sum_v n_v[v] \in \mathrm{Div}(C)$, where
- $n_v := \begin{cases} 0 & \text{good reduction,} \\ 1 & \text{multiplicative,} \\ 2 & \text{additive.} \end{cases}$

Szpiro ratio:

- $\sigma := \dfrac{\deg(\Delta)}{\deg(N)}$
- Pesenti-Szpiro:

$$\deg(\Delta) \leq 12(g-1) + 6\deg(N),$$

so

$$\sigma = O(g),$$

$$\sigma \leq 6 \quad \text{if } g \leq 1.$$

$$h_x(P) = \tfrac{1}{2}\deg(x(P)),$$

$$\widehat{h}(P) = \lim_{n\to\infty} \frac{1}{n^2} h(nP),$$

$$\Delta_v = -16(4A^3 + 27B^2),$$

$$\Delta = \sum_v v(\Delta_v)[v]$$

$$\neq 0.$$

Results

	$\widehat{h}(P) \geq \frac{\deg(\Delta)}{\cdots}$
Hindry-Silverman (1988)	$10^{12.6}$ or $10^{12.1+23g}$
Elkies	50656 if $\sigma \leq 6$ (e.g. if $g \leq 1$)
Naskręcki-Streng	$285\ \sigma^5$ and $10^9(g-1)^3$ if $g \geq 2$

$$\sigma = \frac{\deg(\Delta)}{\deg(N)} = O(g).$$

First idea (Hindry-Silverman)

Idea:
For $\omega = (\omega_1, \omega_2, \omega_3, \ldots) \in \mathbb{R}^{\infty}_{\geq 0}$, consider

$$S := \left(\sum_n n^2 \omega_n\right) \widehat{h}(P) = \sum_{n=1}^{\infty} \omega_n \widehat{h}(nP) \geq \sum_{\substack{v \\ \text{bad}}} \sum_n \omega_n c_v(nP).$$

Tool (Néron local height). If $Q \neq O$, then

- $\widehat{h}(Q) = \sum_v \widehat{h}_v(Q)$,
- $\widehat{h}_v(Q) = \max\left\{0, -\frac{1}{2} v(x(Q))\right\} + c_v(Q)$
  $\geq c_v(Q)$,
- $c_v(Q) = 0$ if Q has good reduction.

Correction terms

For simplicity: only multiplicative reduction.
Then

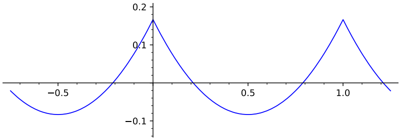
- $c_v(P) = \frac{1}{2}\, b\, B\left(\frac{a}{b}\right)$, where
- $b = b_v = v(\Delta) > 0$,
- $a = a_v :=$ (component distance of P to 0),

Then

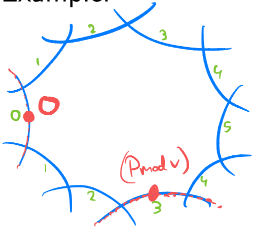
$$S := \left(\sum_n n^2 \omega_n\right) \widehat{h}(P) \geq \sum_{\substack{v \\ \text{bad}}} \sum_n \omega_n c_v(nP)$$

$$\geq \frac{1}{2} \sum_{\substack{v \\ \text{bad}}} b_v \sum_n \omega_n B\left(n\, \frac{a_v}{b_v}\right).$$

Periodic Bernoulli polynomial
$B(t) = \{t\}^2 - \{t\} + \frac{1}{6}$,



Example:



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$c_v(P) = \frac{10}{2} B\left(\frac{3}{10}\right)$
≈ -0.217

Second idea (Elkies)

Suppose $s_1, s_2 \in \mathbb{R}_{\geq 0}$ are such that $\forall b \in \mathbb{Z}_{\geq 1}$ and $\forall a \in (\mathbb{Z}/b\mathbb{Z})$:

$$(\star) \qquad \frac{1}{2}\sum_n \omega_n B\left(n\frac{a}{b}\right) \geq \frac{s_2}{b} - s_1.$$

$$b_v = v(\Delta),$$

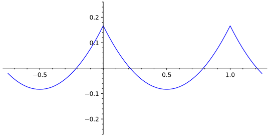
$$B(x) = \{x\}^2 - \{x\} + \frac{1}{6},$$

Then

$$\left(\sum_n n^2\omega_n\right)\widehat{h}(P) \geq \frac{1}{2}\sum_{\substack{v\\ \text{bad}}} b_v \sum_n \omega_n B\left(n\,\frac{a_v}{b_v}\right) \geq \sum_{\substack{v\\ \text{bad}}} s_2 - s_1 b_v$$

$$= s_2 \deg(N) - s_1 \deg(\Delta)$$

$$= \left(\frac{s_2}{\sigma} - s_1\right)\deg(\Delta).$$



Elkies minimizes $(\sum_n n^2\omega_n)$ subject to:

$\omega \in \mathbb{R}^{30}_{\geq 0}$, $(\frac{1}{6}s_2 - s_1) = 1$, and $(\star)$ for all $a, b \leq 60$.

This is a linear optimization problem!

Triangle vectors

Given $m, M \in \mathbb{Z}_{\geq 1}$, define $\tau(m, M) \in \mathbb{R}_{\geq 0}^{\infty}$ by

$$\tau(m, M)_n = \begin{cases} 12(1 - \frac{k}{M+1}) & \text{if } n = km \text{ with } k \leq M, \\ 0 & \text{otherwise.} \end{cases}$$

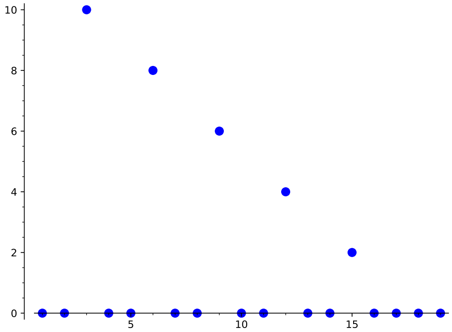
Hindry-Silverman use Fourier analysis to prove

$$\sum_n \tau(m, M)_n B(nt) \geq -1 \quad \text{for all } t \in \mathbb{R}.$$

Observe

$$\sum_n \tau(m, M)_n B\left(n\frac{a}{b}\right) = M \quad \text{if } b \mid m.$$

The proof of Hindry-Silverman uses $\omega = \tau(m, M)$ for some m and M.



graph of $n \mapsto \tau(3, 5)_n$

Our work

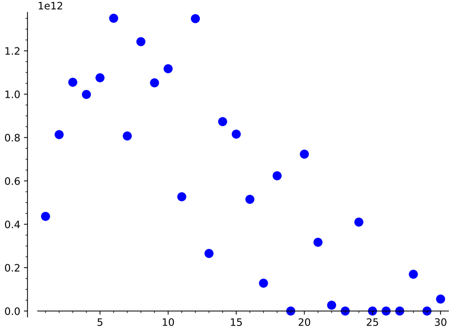
Observation: Elkies's optimal ω for $\sigma \leq 6$ looks like it could be a linear combination of triangle vectors with small m.

The same happens for $\sigma \leq 7$, $\sigma \leq 8$, $\sigma \leq 9$, etcetera.

We find a general solution to the inequalities for every σ, using linear combinations of triangle vectors:

$$\omega = \sum_{b=1}^{c} \left(\frac{\frac{1}{b} - \frac{1}{c+1}}{M_b + 1} \right) \tau(b, M_b), \quad \text{where}$$

$$c = \left\lfloor \frac{5}{2}\sigma \right\rfloor, \qquad M_b = \lceil f b^{-2/3} \rceil - 1, \qquad f \approx 4.14\sigma^{5/3}.$$



graph of $n \mapsto \omega_n$ for Elkies's ω

Our work

Theorem (Naskręcki-Streng)

$$\widehat{h}(P) \geq \frac{1}{285\ \sigma^5} \deg(\Delta)$$

Proof.

$$\omega = \sum_{b=1}^{c} \left(\frac{\frac{1}{b} - \frac{1}{c+1}}{M_b + 1} \right) \tau(b, M_b), \quad \text{where}$$

$$c = \left\lfloor \frac{5}{2}\sigma \right\rfloor,$$

$$M_b = \lceil f b^{-2/3} \rceil - 1,$$

$$f \approx 4.14\ \sigma^{5/3}$$

plus estimates, plus work for arbitrary reduction types and base fields. □

Theorem (Naskręcki-Streng)

$$\widehat{h}(P) \geq \frac{1}{10^9\ (g-1)^3} \deg(\Delta) \quad \text{if } g \geq 2.$$

Proof.

$$\omega = \sum_{b \in \mathcal{B}} \left(\frac{1}{b(M_b + 1)} \right) \tau(b, M_b), \quad \text{where}$$

$$\mathcal{B} = \{ b_v : v \text{ bad} \},$$

$$M_b = \lceil f b^{-2/3} \rceil - 1,$$

$$f \approx 2.25\ \sigma\ (\#\mathcal{B})^{2/3}.$$

plus estimates, plus work for arbitrary reduction types and base fields. □

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17th Algorithmic Number Theory Symposium

- Conference: 6 – 10 July 2026
  Groningen, the Netherlands
- Accepted paper $\Leftrightarrow$ contributed talk

Tentative information:

- Deadline for paper submission 17 January 2026
- Accepted papers are published in
  Research in Number Theory (Springer)
  (same as in 2022 and 2024)

https://antsmath.org

Results

	$[K:\mathbb{Q}] = d$ $\widehat{h}(P) \geq \frac{\log(\mathrm{Norm}(\Delta))}{\cdots}$	function field K $\widehat{h}(P) \geq \frac{\deg(\Delta)}{\cdots}$
Hindry-Silverman (1988)	$(20\sigma)^{8d}10^{1.1+4\sigma}$	$10^{12.6+23g}$
Elkies		50656 if $\sigma \leq 6$ (e.g. if $g \leq 1$)
Petsche (2006)	$O(d^3\sigma^6\log(d\sigma)^2)$	would give $O(\sigma^6)$ and $O(g^6)$
Naskręcki-Streng	?	285 σ^5 and $10^9(g-1)^3$ if $g \geq 2$

$$\sigma = \frac{\deg(\Delta)}{\deg(N)} = O(g).$$