$\mathfrak{a}$-MULTIPLICATIONS

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CONTENTS

1. DEFINITION AND BASIC PROPERTIES

Let k be a field, E a CM-algebra (that is, a product of CM-fields), and $\mathcal{O}_E$ the maximal order of E. The objects that we hope to classify in this note are given in the following

Definition 1.1. *An E-*principal pair *is a pair (X, ι), where X is an abelian variety of dimension* $\deg(E|\mathbf{Q})/2$ *over k and $\iota : E \to \mathbf{Q} \otimes \mathrm{End}(X)$ is an injective homomorphism such that $\iota(\mathcal{O}_E) \subset \mathrm{End}(X)$. A* morphism of E-principal pairs *$(X, \iota) \to (X', \iota')$ is a homomorphism of abelian varieties $\varphi : X \to X'$ satisfying $\varphi\iota(r) = \iota'(r)\varphi$ for every r in $\mathcal{O}_E$.*

Certainly there exist CM abelian varieties, even simple ones, that do not admit such an ι. However, the main theorems of complex multiplication for such abelian varieties actually follow from the principal case. So we have not unduly restricted ourselves.

The reason for introducing principal pairs is that only the presence of ι allows us to unambiguously define the type of (X, ι). We also have

Proposition 1.2. *Let (X, ι) be an E-principal pair. Then* $\mathrm{End}(X, \iota) = \iota(\mathcal{O}_E)$.

Proof. This follows from the statement that $\mathbf{Q} \otimes \iota(\mathcal{O}_E)$ is its own commutant in $\mathbf{Q} \otimes \mathrm{End}(X)$, which was in Peter's talk. □

Note that the inclusion $\mathrm{End}(X, \iota) \subset \mathrm{End}(X)$ can be strict, though not in characteristic 0.

Definition 1.3 ([Mil], 7.17)**.** *Let (X, ι) be an E-principal pair, and let $\mathfrak{a}$ be an ideal of $\mathcal{O}_E$. An* $\mathfrak{a}$-multiplication *(of (X, ι)) is a surjective homomorphism of abelian varieties*

$$\lambda_{\mathfrak{a}}^X : X \longrightarrow X_{\mathfrak{a}}$$

satisfying the following properties:

(i) *For every $a \in \mathfrak{a}$, the endomorphism $\iota(a) : X \to X$ factors (uniquely) through $\lambda_{\mathfrak{a}}^X$;*
(ii) *$\lambda_{\mathfrak{a}}^X$ is universal with the property in (i), meaning that it factors (uniquely) through any homomorphism $X \to Z$ also enjoying it.*

The properties above determine $\lambda_{\mathfrak{a}}^X$ up to (unique) isomorphism in the sense that if $\lambda_{\mathfrak{a}}^X : X \to X_{\mathfrak{a}}$ and $\lambda_{\mathfrak{a}}'^X : X \to X'_{\mathfrak{a}}$ are $\mathfrak{a}$-multiplications, then there exists a (unique) isomorphism of abelian varieties $\varphi : X_{\mathfrak{a}} \to X'_{\mathfrak{a}}$ such that $\varphi\lambda_{\mathfrak{a}}^X = \lambda_{\mathfrak{a}}'^X$.

Proposition 1.4. *Let (X, ι) be an E-principal pair, and let $\mathfrak{a}$ be an ideal of $\mathcal{O}_E$. Then there exists an $\mathfrak{a}$-multiplication of (X, ι).*

Proof. Choose a_1 and a_2 such that $\mathfrak{a} = (a_1, a_2)$. Consider the homomorphism $X \to X^2$ given by sending $x \mapsto (a_1 x, a_2 x)$. One checks that this homomorphism works (or see Proposition 7.20 in [Mil]): indeed, an analogous construction works for any set of generators of $\mathfrak{a}$. □

For a principal ideal (a) of $\mathcal{O}_E$, the endomorphism $a : X \to X$ is an (a)-multiplication. An $\mathfrak{a}$-multiplication is an isogeny if and only if $\mathfrak{a}$ is an invertible ideal of $\mathcal{O}_E$. There is then a (unique) pair $(X_{\mathfrak{a}}, \iota_{\mathfrak{a}})$ for which $\lambda_{\mathfrak{a}}$ is a morphism of pairs. In terms of the concrete $\mathfrak{a}$-multiplication defined above, we simply have

$$\iota_{\mathfrak{a}}(r)(a_1 x, a_2 x) = (r a_1 x, r a_2 x).$$

Note: During the talk, I claimed that a similar construction could be performed for arbitary isogenies. **This is dead wrong.**

We call this pair $(X_{\mathfrak{a}}, \iota_{\mathfrak{a}})$ the $\mathfrak{a}$-*transform* of (X, ι). Note that there are canonical isomorphisms $(X, \iota) \cong (X_{(a)}, \iota_{(a)})$.

Proposition 1.5. *Let (X, ι) be as above and let $\mathfrak{a}$ and $\mathfrak{b}$ be invertible ideals in $\mathcal{O}_E$. Then we have:*

(i) $\lambda_{\mathfrak{b}}^{X_{\mathfrak{a}}} \lambda_{\mathfrak{a}}^X$ *is canonically isomorphic to* $\lambda_{\mathfrak{b}\mathfrak{a}}^X$.
(ii) $\deg(\lambda_{\mathfrak{a}}^X) = \mathrm{Nm}(\mathfrak{a}) = [\mathcal{O}_E : \mathfrak{a}]$.
(iii) $\mathfrak{a} \subset \mathfrak{b}$ *if and only if there exists a homomorphism* $\varphi : X_{\mathfrak{a}} \to X_{\mathfrak{b}}$ *with* $\varphi \lambda_{\mathfrak{b}}^X = \lambda_{\mathfrak{a}}^X$. *This homomorphism is then a morphism of E-principal pairs.*
(iv) *We have* $\mathrm{Hom}((X, \iota), (X_{\mathfrak{a}}, \iota_{\mathfrak{a}})) = \lambda_{\mathfrak{a}}^X \iota(\mathfrak{a}^{-1})$. *In particular, all these homomorphism are multiplication by some ideal.*

Proof. (i): This can be proved using the universal property, but it is most easily seen using the proof of Proposition 1.4: if $\{a_i\}_i$ is a set of generators for $\mathfrak{a}$ and $\{b_j\}_j$ is a set of generators for $\mathfrak{b}$, then $\{b_j a_i\}_{j,i}$ is a set of generators for $\mathfrak{b}\mathfrak{a}$.

(ii): First let $\mathfrak{a} = (a)$ be principal. Then the degree of $\lambda_{(a)}$ is just the degree of a. $E \subset \mathrm{End}(A)$ acts faithfully and $\mathbf{Q}_l$-linearly on the Tate module $V_l(X)$, and by a standard result on abelian varieties, the degree of an $a \in \mathcal{O}_E \subset E$ can be recovered as the determinant of the linear map $V_l(X) \to V_l(X)$ induced by a. By considering the dimension of this vector space over $\mathbf{Q}_l$ and using faithfulness, we see that in fact V_l is a free E_l-module of rank 1. So the determinant in question is equal to the determinant of a acting on E_l. Since determinants do not change under base extension, this is simply the determinant of a acting on E. But this is equal to $\mathrm{Nm}(a)$ ("well-known").

Now let $\mathfrak{a}$ be an arbitrary ideal in $\mathcal{O}_E$. Take an ideal $\mathfrak{b}$ relatively prime to $\deg(\lambda_{\mathfrak{a}})$ such that $\mathfrak{b}\mathfrak{a} = (a)$ is principal. Then by (i) and multiplicativity of degree and norm:

$$\deg(\lambda_{\mathfrak{b}}^{X_{\mathfrak{a}}})\deg(\lambda_{\mathfrak{a}}^X) = \deg(\lambda_{\mathfrak{b}\mathfrak{a}}^X) = \deg(\lambda_{(a)}^X) = \mathrm{Nm}(a) = \mathrm{Nm}(\mathfrak{b}\mathfrak{a}) = \mathrm{Nm}(\mathfrak{b})\mathrm{Nm}(\mathfrak{a}).$$

By coprimeness $\deg(\lambda_{\mathfrak{a}}^X) | \mathrm{Nm}(\mathfrak{a})$. Similiarly, $\deg(\lambda_{\mathfrak{b}}^{X_{\mathfrak{a}}}) | \mathrm{Nm}(\mathfrak{b})$, so we must in fact have equality.

(iii): If $\mathfrak{a} \subset \mathfrak{b}$, then necessarily $\varphi = \lambda_{\mathfrak{a}\mathfrak{b}^{-1}}^{X_{\mathfrak{a}}}$. Conversely, if such a φ exists, then one checks formally that $\lambda_{\mathfrak{b}}^X$ satisfies the properties in the definition of $\lambda_{\mathfrak{a}+\mathfrak{b}}^X$. Certainly $\mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$, but then we must have equality considering the degrees of $\lambda_{\mathfrak{b}}^X \cong \lambda_{\mathfrak{a}+\mathfrak{b}}^X$. So indeed $\mathfrak{a} \subset \mathfrak{b}$.

(iv) See 3.2.4 in [Lan83] . □

$\mathfrak{a}$-multiplications form a broad class of morphisms:

Proposition 1.6. *Let $\varphi : (X, \iota) \to (X', \iota')$ be a separable isogeny of pairs. Then there exists an ideal $\mathfrak{a}$ of $\mathcal{O}_E$ such that φ is an $\mathfrak{a}$-multiplication.*

Proof. Given φ, consider the annihilator ideal of the $\mathcal{O}_E$-module $\ker(\varphi)(\bar{k})$:

$$\mathfrak{a} = \{a \in \mathrm{End}(X) : a|_{\ker(\varphi)(\bar{k})} = 0\}.$$

Decomposing $\ker(\varphi)(\bar{k})$ as a sum of modules on which $\mathcal{O}_E$ acts transitively and taking the intersection of the annihilators of these modules, we see that this ideal has index at most $|\ker(\varphi)(\bar{k})|$ in $\mathcal{O}_E$. On the other hand, f factors through all a in $\mathfrak{a}$ since these a vanish on $\ker(\varphi)(\bar{k})$, hence by separability of φ on $\ker(\varphi)$. So by the universal property, $\lambda_{\mathfrak{a}}$ factors through φ. So by considering degrees, we see that $[\mathcal{O}_E : \mathfrak{a}]$ in fact equals $|\ker(\varphi)(\bar{k})|$, and that $\varphi \cong \lambda_{\mathfrak{a}}$. □

Of course there can also exist inseparable isogenies that are $\mathfrak{a}$-multiplications: consider multiplication by $\mathrm{char}(k)$, for example. In fact, I don't know an isogeny of pairs that is not an $\mathfrak{a}$-multiplication, nor I am sure that the above argument cannot be generalized to the non-separable case. The point is that there doesn't seem to be a natural finite $\mathcal{O}_E$-module showing up in the general case except the one in the proof, and that one doesn't cut it. Incidentally, a result that René will prove will show that for elliptic curves, any isogeny of pairs is an $\mathfrak{a}$-multiplication.

Theorem 1.7. *Let (X, ι) be an E-principal pair, and let $\mathfrak{a}$ and $\mathfrak{a}'$ be invertible ideals of $\mathcal{O}_E$. Then $(X_{\mathfrak{a}}, \iota_{\mathfrak{a}})$ is isomorphic to $(X_{\mathfrak{a}'}, \iota_{\mathfrak{a}'})$ if and only if $\mathfrak{a}^{-1}\mathfrak{a}'$ is principal, and one has an isomorphism of $\mathcal{O}_E$-modules*

$$\mathrm{Hom}((X_{\mathfrak{a}}, \iota_{\mathfrak{a}}), (X_{\mathfrak{a}'}, \iota_{\mathfrak{a}'})) \cong (\mathfrak{a}^{-1}\mathfrak{a}')^{-1}.$$

In other words, the set of $\mathfrak{a}$-transforms of (X, ι) is a $\mathrm{Cl}(E)$-torsor in a natural way.

Proof. The previous Proposition and the remark before it imply that we indeed have an action of $\mathrm{Cl}(E)$ on the set of isomorphism classes, so we need only determine for which invertible $\mathfrak{b}$ we have that $(X_{\mathfrak{b}}, \iota_{\mathfrak{b}})$ is isomorphic to (X, ι). We have seen that this is true for principal $\mathfrak{b}$. Conversely, if these pairs are isomorphic, then $\lambda_{\mathfrak{b}} \cong r = \lambda_{(r)}$ for some $a \in \mathrm{End}(X, \iota)$. But then $\mathfrak{b} = (r)$ by Proposition 1.5(iii). □

2. The case $k = \mathbf{C}$: classification of E-principal pairs

Now fix a CM-type Φ of E. Then David has shown us that all E-principal pairs (X, ι) of type (E, Φ) over $\mathbf{C}$ have the property that the associated complex Lie groups $(X^{\mathrm{an}}, \iota^{\mathrm{an}})$ are of the form $\mathbf{C}^n/\Phi(\mathfrak{a})$, where $\mathfrak{a}$ is an invertible ideal of $\mathcal{O}_E$ and $r \in \mathcal{O}_E$ acts by $\mathbf{R}$-linearly extending its action on $\mathfrak{a}$.

Homomorphisms from $\mathbf{C}^n/\Phi(\mathfrak{a})$ to $\mathbf{C}^n/\Phi(\mathfrak{a}')$ lift to linear maps $\mathbf{C}^n \to \mathbf{C}^n$ and can be identified with elements of $\mathfrak{a}^{-1}\mathfrak{a}'$ under Φ: isogenies with invertible elements. Note that any such isogeny is in fact a morphism of principal pairs since E acts on the two tangent spaces at 0 in the same way.

Theorem 2.1. *Let (X, ι) be an E-principal pair over $\mathbf{C}$ of type (E, Φ), corresponding to an invertible ideal $\mathfrak{a}$ of $\mathcal{O}_E$. Then every other pair (X', ι') of type (E, Φ) can be obtained from (X, ι) by multiplication by some ideal.*

The set of isomorphism classes of E-principal pairs of type Φ over $\mathbf{C}$ is a $\mathrm{Cl}(E)$-torsor in a natural way.

Proof. For the first part, suppose that (X, ι) and (X', ι') correspond to $\mathbf{C}^n/\Phi(\mathfrak{a})$ and $\mathbf{C}^n/\Phi(\mathfrak{a}')$, respectively. Choose an invertible element $r \in \mathfrak{a}^{-1}\mathfrak{a}'$. Then $\Phi(r)$ descends to an isogeny of principal pairs $(X, \iota) \to (X', \iota')$ that is a multiplication by an ideal considering Proposition 1.6.

(In fact, by the proof of that Proposition, the ideal that works is the annihilator of the kernel of the map induced by r. This kernel is given isomorphic to the $\mathcal{O}_E$-module $r^{-1}\mathfrak{a}'/\mathfrak{a} \subset E/\mathfrak{a}$, and the corresponding annihilator is $r\mathfrak{a}'^{-1}\mathfrak{a}$.)

The rest now follows from Theorem 1.7. □

REFERENCES

[Lan83] Serge Lang, *Complex multiplication*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 255, Springer-Verlag, New York, 1983. MR MR713612 (85f:11042)

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