EXISTENCE AND CLASSIFICATION OF CM ABELIAN VARIETES OVER $\mathbb{C}$.

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Let $V \cong C^g$ be a complex vector space, and let $\Lambda \subset V$ be a lattice.

Definition. A *Riemann form* on V/Λ is the *imaginary part* of a positive definite Hermitian form H on V such that $H(\Lambda) \subset \mathbb{R} + i\mathbb{Z}$.

Equivalently, a Riemann form is a map $E : V \times V \to \mathbb{R}$ such that:

(a) E is $\mathbb{R}$-bilinear,
(b) E is alternating,
(c) $E(u, v) \in \mathbb{Z}$ for $u, v \in \Lambda$,
(d) $(u, v) \mapsto E(iu, v)$ is positive definite and symmetric,

The form H of the polarizations lecture is obtained by $H(u, v) = E(iu, v) + iE(u, v)$. This H is positive definite if and only if $E(iu, v)$ is positive definite.

Recall that H is positive definite if and only if the line bundle $L(H, \alpha)$ (from the polarizations lecture) is ample. Furthermore, a complex manifold M admits an ample line bundle if and only if M is projective, and any projective complex manifold is an algebraic variety. Thus complex abelian varieties are exactly complex tori that admit a Riemann form.

Some notation: let K be a CM field and $\Phi = (\phi_1, \ldots, \phi_g)$ be a CM type on K. Then there is an isomorphism of $\mathbb{R}$-algebras $\Phi : K \otimes \mathbb{R} \to \mathbb{C}^g$ defined by setting

$$x \otimes r \quad \mapsto \quad (r\phi_1(x), \ldots, r\phi_g(x))$$

and extending linearly to sums of tensors.

Theorem 1 (Existence)**.** *Let K be a CM field of degree $2g$. For any CM type Φ of K, any fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$, and any $\xi \in K$ satisfying*

(1) $\xi^2 \ll 0$ *(i.e., ξ is totally imaginary),*
(2) $\operatorname{Im} \phi_i(\xi) > 0$ *for all i,*
(3) $\xi \in (\mathfrak{a}\bar{\mathfrak{a}})^\vee = (\mathfrak{a}\bar{\mathfrak{a}}\mathfrak{D})^{-1}$, *where*
   - $^\vee$ *indicates the trace dual,*
   - $\mathfrak{D}$ *is the different of K,*
   - *the equality holds since $\mathcal{O}_K^\vee = \mathfrak{D}^{-1}$,*

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there exits an abelian variety that has CM by $\mathcal{O}_K$, CM type Φ, and a polarization given by the Riemann form defined by

$$\begin{array}{rcl} E : K \times K & \to & \mathbb{Q} \\ (\Phi(u), \Phi(v)) & \mapsto & \mathrm{Tr}_{K/\mathbb{Q}}(\xi \bar{u} v) \end{array}$$

for $u, v \in K$, extended $\mathbb{R}$-linearly to $\mathbb{C}^g \times \mathbb{C}^g$.

Proof. Let $A = \mathbb{C}^g/\Phi(\mathfrak{a})$. Since $\Phi(\mathfrak{a})$ is a full-rank lattice in $\mathbb{C}^g$, the manifold A is a complex torus. We must check:

1. E is a Riemann form on $\mathbb{C}^g/\Phi(\mathfrak{a})$.

   (a) Obvious.
   (b) Since ξ is totally imaginary, we have
   $$E(u, v) = \mathrm{Tr}_{K/\mathbb{Q}}(\xi \bar{u} v) = \mathrm{Tr}_{K/\mathbb{Q}}(\overline{-\xi u \bar{v}}) = -E(v, u)$$
   (c) This is exactly the definition of "trace dual."
   (d) Follows from properties (1) and (2) of ξ. (Trust me...)

2. A has CM by $\mathcal{O}_K$. Define the map

$$\begin{array}{rcl} \iota : \mathcal{O}_K & \to & \mathrm{End}(\mathbb{C}^g/\Phi(\mathfrak{a})) \\ \alpha & \mapsto & \mathrm{diag}\,\Phi(\alpha) := \begin{pmatrix} \phi_1(\alpha) & & 0 \\ & \ddots & \\ 0 & & \phi_g(\alpha) \end{pmatrix} \end{array}$$

This defines the action of $\mathcal{O}_K$ on $\mathbb{C}^g$; since $\mathfrak{a}$ is a (fractional) ideal we have $\mathcal{O}_K \mathfrak{a} = \mathfrak{a}$, and thus the action factors through the quotient.

3. A has CM type Φ. To determine the CM type we look at the action of $\mathcal{O}_K$ induced by ι on the tangent space at the origin,

$$\iota : \mathcal{O}_K \to \mathrm{End}(A) \to \mathrm{End}(T_0 A) = \mathrm{End}(\mathbb{C}^g) = \mathrm{Mat}_{n \times n}(\mathbb{C})$$

When this action is diagonalized the CM type $\Psi = (\psi_1, \ldots, \psi_g)$ of A is defined by setting $\psi_i(\alpha)$ to be the ith diagonal entry of the resulting diagonal matrix. With our definition of ι above, this action is already diagonal, and it is easy to see that $\psi_i(\alpha) = \phi_i(\alpha)$, so $\Psi = \Phi$. □

Remark. The conditions of Theorem 1 are not vacuous; namely, there always exists a $\xi \in K$ with properties (1)–(3). Given an $\alpha \in K$, we let $\beta = \alpha - \bar{\alpha}$; then β is totally imaginary. We can then choose a unit $\gamma \in \mathcal{O}_{K_0}^*$ such that $\mathrm{Im}\,\phi_i(\gamma\beta) > 0$ for all i. Finally, we choose $n \in \mathbb{Z}$ such that $n\gamma\beta \in (\mathfrak{a}\bar{\mathfrak{a}})^\vee$, and set $\xi = n\gamma\beta$.

Definition. A CM-type Φ of K is *induced* from a CM-subfield $K' \subset K$ if it is of the form $\Phi = \{\phi : \phi|_{K'} \in \Phi'\}$ for some CM-type Φ' of K'. We call Φ *primitive* if it is not induced from a proper subfield of K.

Theorem 2 (Classification)**.** *Let (A, ι) be an abelian variety over $\mathbb{C}$ with CM by $\mathcal{O}_K$ and primitive CM type Φ. Then there exists a fractional ideal $\mathfrak{a}$ and an isomorphism*

$\theta : A \xrightarrow{\sim} \mathbb{C}^g/\Phi(\mathfrak{a})$ *such that the following diagram commutes for every* $\alpha \in \mathcal{O}_K$*:*

$$\begin{array}{ccc} A & \xrightarrow{\theta} & \mathbb{C}^g/\Phi(\mathfrak{a}) \\ {\scriptstyle\iota(\alpha)}\downarrow & & \downarrow{\scriptstyle(\operatorname{diag}\Phi)(\alpha)} \\ A & \xrightarrow{\theta} & \mathbb{C}^g/\Phi(\mathfrak{a}) \end{array} .$$

Furthermore, if p *is a polarization on* A*, and* E *is a Riemann form on* $\mathbb{C}^g/\Phi(\mathfrak{a})$ *corresponding to* p *via* θ*, then there exists a* $\xi \in K$ *satisfying (1)–(3) above such that* E *is given as in Theorem 1.*

Sketch of proof: Let $\theta : A \to T_0A/\Lambda$ be the isomorphism from Richard's talk (i.e., the inverse of the exponential map). The map $\iota : \mathcal{O}_K \to \operatorname{End}(A)$ induces an action of $\mathcal{O}_K$ on T_0A and thus also on Λ. Since A has CM type Φ, we can choose a basis of $T_0A \cong \mathbb{C}^g$ such that $\mathcal{O}_K$ acts diagonally. If we choose a K-basis for $\Lambda \otimes \mathbb{Q}$, then we can identify $\Lambda \otimes \mathbb{Q}$ with K. Thus Λ is a nonzero $\mathcal{O}_K$-submodule of K, i.e., a fractional ideal $\mathfrak{a}$.

Let E be a Riemann form as in the statement. For any $u, v, x \in K$, the map $\xi_{u,v} : x \mapsto E(\Phi(x)\Phi(u), \Phi(v))$ is $\mathbb{Q}$-linear, so there is a function $\omega(u, v)$ such that $\xi_{u,v}(x) = \operatorname{Tr}_{K/\mathbb{Q}}(x\omega(u, v))$.[1] Furthermore, one can use the properties (a)–(d) of E to show that ω is u-linear, v-anti-linear, and alternating, which implies that it is of the form $w(u, v) = \xi u\bar{v}$ for some ξ satisfying properties (1) and (3). Property (2) follows from positive definiteness of E. Thus $E(\Phi(u), \Phi(v)) = \operatorname{Tr}_{K/\mathbb{Q}}(\xi u\bar{v})$. □

If ξ is as in the construction above, then we have

(1) $$\deg p := \#\ker p = [(\mathfrak{a}\bar{\mathfrak{a}}\mathcal{D})^{-1} : \xi\mathcal{O}_K].$$

Definition. Let (A, ι, p) and (A', ι', p') be polarized abelian varieties together with a CM structure. An isomorphism (of varieties) $f : A \to A'$ is an *isomorphism between polarized abelian varieties* (A, ι, p) and (A', ι', p') if the following two diagrams commute for all $\alpha \in \mathcal{O}_K$:

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle\iota(\alpha)}\downarrow & & \downarrow{\scriptstyle\iota'(\alpha)} \\ A & \xrightarrow{f} & A' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \hat{A} & \xleftarrow[\hat{f}]{} & \hat{A}' \end{array}$$

Let K be a CM field, and let

$$S(K) = \left\{ (A, \iota, p) : \begin{array}{c} A/\mathbb{C} \text{ an abelian variety,} \\ \iota : \mathcal{O}_K \hookrightarrow \operatorname{End}(A), \\ p \text{ a polarization} \end{array} \right\} / \cong,$$

and let $S_d(K) = \{(A, \iota, p) \in S(K) : \deg p = d\}$.

[1] I'm sweeping things under the rug here — this is where we use the primitive hypothesis on Φ, to guarantee that E is what Lang calls "Φ-admissible."

Let

$$T(K) = \left\{ (\mathfrak{a}, \Phi, \xi) : \begin{array}{c} \mathfrak{a} \subset \mathcal{O}_K \text{ a fractional ideal,} \\ \Phi \text{ a CM type of } K, \\ \xi \in \mathcal{O}_K \text{ satisfying (1)–(3).} \end{array} \right\},$$

and let

$$T_d(K) = \{(\mathfrak{a}, \Phi, \xi) \in T(K) : [(\mathfrak{a}\bar{\mathfrak{a}}\mathcal{D})^{-1} : \xi\mathcal{O}_K] = d\}.$$

Theorem 3 (Finiteness)**.** *For any CM field K and any $d \in \mathbb{Z}_{>0}$, $S_d(K)$ is finite.*

Proof sketch: By Theorem 1, there is a map $\chi : T(K) \to S(K)$. By Theorem 2, the map χ is surjective. By (1) the map χ induces a surjection from $T_d(K)$ to $S_d(K)$. We claim that that there is a finite set $U_d(K) \subset T_d(K)$ such that $\chi : U_d(K) \to S_d(K)$ is surjective.

First, there are finitely many CM types on K, so the number of possible Φ is already finite.

Second, for $u \in K$, the map $x \mapsto \Phi(u)x$ gives an isomorphism $\chi(\mathfrak{a}, \Phi, \xi) \cong \chi(u\mathfrak{a}, \Phi, (u\bar{u})^{-1}\xi)$. Thus χ remains surjective when we restrict the possible $\mathfrak{a}$ to contain one representative of each ideal class of $\mathcal{O}_K$, which is a finite set.

Third, for any given $\mathfrak{a}$ there are finitely many ideals $\xi\mathcal{O}_K$ with $[(\mathfrak{a}\bar{\mathfrak{a}}\mathcal{D})^{-1} : \xi\mathcal{O}_K] = d$. For each such ideal, pick (if possible) a generator ξ_0 with properties (1)–(3). Any ξ with the desired properties is equal to $v\xi_0$ for some totally positive $v \in \mathcal{O}_{K_0}^*$. (Totally positive by property (1), a unit because degree = d.)

Suppose v, v' are totally positive in $\mathcal{O}_{K_0}^*$. If there is some $u \in \mathcal{O}_K^*$ such that $u\bar{u}v\xi_0 = v'\xi_0$, then $\chi(\mathfrak{a}, \Phi, v\xi_0) \cong \chi(\mathfrak{a}, \Phi, v'\xi_0)$. In other words, if $v'/v \in N_{K/K_0}(\mathcal{O}_K^*)$, then the polarizations defined by v and v' are isomorphic. Thus we can fix ξ_0 and restrict the choice of ξ in the domain of χ to ξ_0 times coset representatives of $\mathcal{O}_{K_0}^*/N_{K/K_0}(\mathcal{O}_K^*)$, and χ remains surjective. But this quotient is finite (the norm group contains $(\mathcal{O}_{K_0}^*)^2$), so we have restricted to a finite set of ξ. □

Theorem 4 (Algebraicity)**.** *Any polarized abelian variety over $\mathbb{C}$ with CM by $\mathcal{O}_K$ is isomorphic to one defined over a number field.*

Hand-wavy argument: Consider the (coarse?) moduli space $\mathcal{M}$ of polarized abelian varieties over $\mathbb{C}$. Let $(A, \iota, p) \in S_d(K)$ (where $d = \deg p$). Since $S_d(K)$ is finite, we can choose an affine open $U \subset \mathcal{M}$ containing all the points corresponding to elements of $S_d(K)$ and write the coordinates of the point corresponding to (A, ι, p) as $(j_1(A, \iota, p), \ldots, j_n(A, \iota, p))$. Now form the polynomials

$$H_i(x) = \prod_{(A,\iota,p) \in S_d(K)} (x - j_i(A, \iota, p))$$

for $i = 1, \ldots, n$. Now any $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$ (the group of ring automorphisms of $\mathbb{C}$) permutes $S_d(K)$, so $H_i(x)$ is fixed by $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$ and thus has coefficients in $\mathbb{Q}$. Thus the $j_i(A, \iota, p)$ are algebraic, and it follows from some more moduli theory that (A, ι, p) can be defined over a finite extension of $L = \mathbb{Q}(j_1(A, \iota, p), \ldots, j_n(A, \iota, p))$. □

If A is an elliptic curve, then we can make the above precise: we have $n = 1$, j_1 is the j-invariant, and A can be defined over $\mathbb{Q}(j(A))$.

Theorem 5 (Isogeny)**.** *Let $(A, \iota), (B, \jmath)$ be two abelian varieties over $\mathbb{C}$ with CM by $\mathcal{O}_K$. If A and B have the same CM type Φ, then A and B are isogenous.*

Proof. By Theorem 2, we can write $A = \mathbb{C}^g/\Phi(\mathfrak{a})$ and $B = \mathbb{C}^g/\Phi(\mathfrak{b})$. An isogeny from A to B is given by a matrix $M \in \mathrm{Mat}_n(\mathbb{C})$ such that $M\Phi(\mathfrak{a}) \subset \Phi(\mathfrak{b})$. It is clear that if $\alpha \in \mathfrak{b}\mathfrak{a}^{-1}$ then $\operatorname{diag}\Phi(\alpha)$ is such a matrix. □

The isogeny constructed in the above proof is a $\mathfrak{c}$-*multiplication*, where $\mathfrak{c} = \alpha\mathfrak{a}\mathfrak{b}^{-1}$, and $(B, \jmath)$ is a $\mathfrak{c}$-*transform* of (A, ι). We will learn what these terms mean in a future talk.

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