A QUICK GUIDE TO POLARIZATIONS

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CONTENTS

1. THE DUAL ABELIAN VARIETY

Given an elliptic curve (E, O) over a field k, the group (scheme) Pic_E of line bundles on E is an interesting object to consider. In fact, the subgroup Pic_E^0 of line bundles of degree 0 is already very useful to study E, since for every field extension K of k, one has an isomorphism

$$\begin{aligned} E(K) &\xrightarrow{\sim} \mathrm{Pic}_E^0(K) \\ P &\longmapsto \mathcal{O}_E([O] - [P]). \end{aligned} \tag{1.1}$$

Here, $\mathrm{Pic}_E^0(K)$ are those line bundles L of degree 0 on E that can be defined over K, which in the less complicated case of elliptic curves simply means that one of the divisors whose associated line bundle is isomorphic to L is defined over K.

The isomorphism above can be used define the group law on E. In the other direction, it gives an interpretation of Pic_E^0 as a geometric object. One way to generalize this result is to consider the group Pic_C^0 for arbitrary curves, which allows one to construct the geometric object (and abelian variety!) called the *Jacobian* of C. In general, however, an isomorphism resembling (1.1) above then only holds if $C(K)$ is nonempty.

In this section, we will generalize in another direction by considering higher-dimensional abelian varieties instead of curves of higher genus. So let $f : X \to \mathrm{Spec}\, k$ be an abelian variety. We will construct $\mathrm{Pic}_{X|k}$ by taking it to be a scheme representing a certain contravariant functor $\mathscr{F}$, that is, a scheme whose functor of points is naturally isomorphic to $\mathscr{F}$. The question is what functor $\mathscr{F}$ to take. A logical first guess would be the functor

$$\mathscr{P}_{X|k} : T \longmapsto \mathrm{Pic}(X \times_k T),$$

acting on morphisms by using the pullback of line bundles. However, this definition cannot work. For example, take $X = \mathrm{Spec}\, k$ to be the trivial abelian variety over k. Then $\mathscr{P} = \mathscr{P}_{X|k}$ is defined by $\mathscr{P}(T) = \mathrm{Pic}(T)$. Let U_i be an open affine cover of T. If $\mathscr{P}$ were respresentable, then any element of $\mathscr{P}(T)$ would be determined by the elements of $\mathscr{P}(U_i)$ it induces ("gluing sections"). However, since the cohomology of affine varieties is trivial, *any* element of $\mathscr{P}(T)$ gives rise to the same elements in the $\mathscr{P}(U_i)$. Taking T such that $\mathrm{Pic}(T)$ is non-trivial, one sees that $\mathscr{P}_{X|k}$ is not representable.

The simplest solution to this quandary is the following.

Definition 1.1. *Let $X \to \operatorname{Spec} k$ be an abelian variety. A* rigidified line bundle *on $X \times_k T$ is a pair (L, α), where L is a line bundle on $X \times_k T$ and α is an isomorphism $\mathcal{O}_{\operatorname{Spec} k} \xrightarrow{\sim} 0_T^* L$. Here $0_T : T \cong \operatorname{Spec} k \times_k T \to X \times_k T$ is the zero section obtained from the zero element $0 : \operatorname{Spec} k \to X$ by base extension.*

A morphism *of rigidified line bundles $(L_1, \alpha_1) \to (L_2, \alpha_2)$ is a morphism $h : L_1 \to L_2$ of line bundles such that $0_T^*(h)\, \alpha_1 = \alpha_2$.*

We define the contravariant functor $\mathscr{P}_{X|k}^{rig}$ on k-schemes to sets on objects by

$$\mathscr{P}_{X|k}^{rig}(T) = \{\textit{rigidified line bundles } (L, \alpha) \textit{ on } X \times_k T\} / \cong$$

and on morphisms by associating with a k-morphism $g : T_1 \to T_2$ the map

$$\begin{aligned} \mathscr{P}_{X|k}^{rig}(T_2) &\longrightarrow \mathscr{P}_{X|k}^{rig}(T_1) \\ (L, \alpha) &\longmapsto (1_X \times g)^*(L, \alpha) \end{aligned}$$

The following Proposition shows that we can ignore the rigidifications α when, for example, $T = \operatorname{Spec} K$ for K a field:

Proposition 1.2. *Let T be a k-scheme. Then the forgetful map*

$$\begin{aligned} \mathscr{P}_{X|k}^{rig}(T) &\longrightarrow \mathscr{P}_{X|k}(T) \\ (L, \alpha) &\longmapsto L \end{aligned}$$

is injective. It is an isomorphism if $\mathrm{Pic}(T)$ is trivial.

Proof. Let L be a fixed line bundle on $X \times_k T$. Then we have

$$(L, \alpha_1) \cong (L, \alpha_2)$$

for any two rigidifications α_1, α_2 of L. This holds as any two rigidifications differ by an element of $\mathcal{O}_T(T)^\times$, which is contained in (and in fact by properness of f_T equal to) the automorphism group of L. This proves injectivity. If $\mathrm{Pic}(T)$ is trivial, then we can construct an inverse

$$\mathscr{P}_{X|k}(T) \longrightarrow \mathscr{P}_{X|k}^{\mathrm{rig}}(T)$$

by sending a line bundle L to the line bundle $L \otimes f^* 0^* L^{-1}$, canonically rigidified by $\mathcal{O}_T \cong 0^* \mathcal{O}_{X_T} \cong 0^* L \otimes 0^* L^{-1} = 0^* L \otimes 0^* f^* 0^* L^{-1} \cong 0^*(L \otimes f^* 0^* L^{-1})$. $\square$

Rigidified line bundles are worthy of their name in that they have no automorphisms. Indeed, any automorphism h of an $(L, \alpha) \in \mathscr{P}_{X|k}^{\mathrm{rig}}(T)$ is an element of

$$\begin{aligned} \mathrm{Aut}(L, L) &= \mathscr{H}om(L, L)(X_T)^\times \\ &= \mathcal{O}_{X_T}(X_T)^\times \\ &= \mathcal{O}_T(T)^\times \end{aligned}$$

where the third equality follows from the fact that the canonical morphism $\mathcal{O}_T \to (f_T)_* \mathcal{O}_{X_T}$ is an isomorphism by the properness of f. But the demand that $0^*(h)\, \alpha = \alpha$ then forces α to equal $1 \in \mathcal{O}_T(T)^\times$.

The fact that it has no automorphisms means that the functor $\mathscr{P}_{X|k}^{\mathrm{rig}}$ has at least a decent chance of being representable, and it turns out that it is:

Theorem 1.3 (Grothendieck-Murre). *Let $f : X \to \operatorname{Spec} k$ be an abelian variety. Then the functor $\mathscr{P}^{rig}_{X|k}$ is representable by a smooth group scheme $\operatorname{Pic}^{rig}_{X|k}$ over k whose connected components are proper.*

The group structure on $\operatorname{Pic}^{\mathrm{rig}}_{X|k}$ used in this Theorem is given on points by using the tensor product of line bundles (and rigidifications). By construction, there exists a rigidified line bundle (U, α_U) (the *universal bundle*) on $X \times \operatorname{Pic}^{\mathrm{rig}}_{X|k}$ with the following universal property: if (L, α) is a rigidified line bundle on $X \times_k T$, then there exists a unique morphism $f : T \to \operatorname{Pic}^{\mathrm{rig}}_{X|k}$ such that $(L, \alpha) \cong (1_X \times f)^*(U, \alpha_U)$.

Definition 1.4. *Let $X \to \operatorname{Spec} k$ be an abelian variety. We define the* dual abelian variety *X^t to be the connected component of $\operatorname{Pic}^{rig}_{X|k}$ that contains the point corresponding to the trivial line bundle on X.*

X^t is indeed an abelian variety over k. Intuitively, X^t "bookmarks" all the rigidified line bundles on X that can be deformed into the trivial bundle over a connected base.

The universal property allows us to construct a plethora of maps $X \to X^t$. Take a line bundle L on X. We have three maps m, p_1, p_2 from $X \times_S X$ to X, given by multiplication, projection on the first component, and projection on the second component, respectively. We can then construct the *Mumford bundle* of L, which is given by

$$\Lambda(L) = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}.$$

One checks that this sheaf can be rigidified (formal verification). Choose a rigidification α of this sheaf. Then since X is connected, there exists a unique map from X to X^t, traditionally denoted by φ_L, such that

$$(\Lambda(L), \alpha) \cong (1_X \times \varphi_L)^*(U, \alpha_U).$$

The morphism φ_L acts on T-points of X as follows. Let $x \in X(T)$: then to see what φ_L does to it we pull back our isomorphism through $1_X \times x$. Denoting translation over x by t_x, we get

$$\begin{aligned}(1_X \times \varphi_L(x))^*(U, \alpha_U) &= (1_X \times x)^*(1_X \times \varphi_L)^*(U, \alpha_U)\\ &\cong (1_X \times x)^*(\Lambda(L), \alpha)\\ &\cong (t_x L_T \otimes L_T^{-1}, \alpha'),\end{aligned}$$

for some rigidification α'. So on points φ_L acts by sending a point $x \in X(T)$ to the class of $(t_x L_T \otimes L_T^{-1}, \alpha')$ in $\operatorname{Pic}^{\mathrm{rig}}_{X|S}(T)$. Using Proposition 1.2, we see that over a field K, we can remove the rigidification from our considerations, so the action on T-points admits the less prolix description

$$x \longmapsto t_x L_T \otimes L_T^{-1}.$$

By the what we have just seen, φ_L sends $0 \in X(k)$ to $0 \in X^t(k)$. In particular, using the result mentioned earlier by Marco that any morphism between abelian varietiess is a composition of a translation and a homomorphism, it is a homomorphism of abelian varieties (this also follows from the so-called *Theorem of the Square*). We have the following fundamental

Theorem 1.5. *Let L be a line bundle on X.*

(i) *The homomorphism φ_L above is an isogeny if and only if L is* non-degenerate, *in the sense that there are only finitely many $x \in \overline{k}$ such that $t_x L \cong L$, a condition that is verified in particular (but not only) if L is ample;*

(ii) *Two line bundles L and M on X give rise to the same homomorphism if and only if $L \otimes M^{-1} \in X^t(k)$ (a property that is independent of the choice of rigidification).*

2. Duality theory

Let $f : X \to Y$ be an homomorphism of abelian varieties $\operatorname{Spec} k$. Denote the Poincaré sheaves for X and Y by (U_X, α_{U_X}) and (U_Y, α_{U_Y}), respectively. There is then a unique morphism $f^t : Y^t \to X^t$ such that we have an isomorphism of rigidified sheaves

$$(1_X \times f^t)(U_X, \alpha_{U_X}) \cong (f \times 1_{Y^t})^*(U_Y, \alpha_{U_Y}).$$

The morphism f^t is called the *dual morphism* of f. Let y_t denote a point of $Y^t(T)$, and denote the corresponding rigidified line bundle on $Y \times_S T$ by (M_{y^t}, β_{y^t}). Then we have

$$\begin{aligned}(1_X \times f^t(y^t))^*(U_X, \alpha_{U_X}) &= (1_X \times y^t)^*(1_X \times f^t)^*(U_X, \alpha_{U_X})\\ &\cong (1_X \times y^t)^*(f \times 1_{Y^t})^*(U_Y, \alpha_{U_Y})\\ &\cong (f \times 1_T)^*(1_Y \times y^t)^*(U_Y, \alpha_{U_Y})\\ &\cong (f \times 1_T)^*(M_{y^t}, \beta_{y^t}).\end{aligned}$$

So on the level of points, the dual morphism simply corresponds to the restriction of the natural map

$$\begin{aligned}\mathscr{P}^{\mathrm{rig}}_{Y|k}(T) &\longrightarrow \mathscr{P}^{\mathrm{rig}}_{X|k}(T)\\ (M, \beta) &\longmapsto (f \times 1_T)^*(M, \beta)\end{aligned}$$

to $Y^t(T) \subseteq \mathscr{P}^{\mathrm{rig}}_{Y|k}(T)$. It is a homomorphism, hence maps Y^t to X^t.

We now come to duality proper. The restriction of the universal bundle (U, α_U) to $X \times_k X^t$ has as part of its data a rigidification of its pullback under $0_X \times 1_{X^t} : \operatorname{Spec} k \times_k X^t \to X \times X^t$. The first hint of the symmetry that is expressed in the upcoming Theorem is the fact that the pullback under $1_X \times 0_{X^t} : X \times_k \operatorname{Spec} k \to X \times X^t$ is trivial as well: this is true by definition of the zero section of X^t.

So by switching the factors and viewing U as a line bundle on X^t, which is also an abelian variety, we can choose a rigidification of U along the zero section of X^t. After modifying this chosen rigidification by an element of $\mathcal{O}_{\operatorname{Spec} k}(\operatorname{Spec} k)^\times$, we can ensure that the new and the old rigidification give rise to the same trivialisations of the pullback of U under $0_X \times 0_{X^t} : \operatorname{Spec} k \times_k \operatorname{Spec} k \to X \times X^t$. Call the new rigidification α^t_U. X^t has a dual variety X^{tt}, and there is a universal bundle (U^t, α_{U^t}) living on $X^t \times X^{tt}$. Hence we obtain a map of varieties $\kappa_{X|k} : X \to X^{tt}$ which induces an isomorphism

$$(U, \alpha^t_U) \cong (1 \times \kappa_{X|k})^*(U^t, \alpha_{U^t})$$

that can be shown to be a homomorphism of abelian varieties.

Theorem 2.1. *Let $f : X \to \operatorname{Spec} k$ be an abelian variety. Then we have:*

(i) *(Double duality) The $\kappa_{X|k}$ constructed above is an isomorphism;*

(ii) *(Triple duality) We have $\kappa^t_{X|k} = \kappa^{-1}_{X^t|k}$;*

(iii) *For any line bundle L on X, we have $\varphi_L = \varphi^t_L \kappa_{X|k}$.*

This Theorem is a formal consequence of *faithfully flat descent* for line bundles, which is used to prove the crucial fact that for any $f : X \to Y$ one has that $\mathrm{Ker}(f^t)$ is isomorphic to the Cartier dual of $\mathrm{Ker}(f)$, but we use it as a black box. For now, note that it implies that we can "flip our bookmark": not only can all line bundles on X that can be deformed into the trivial bundle be constructed as fibers of U on $X \times X^t$ over points of X^t, but we can also obtain all such line bundles on X^t by taking fibers over points of X.

From now on, we will identify X and X^{tt} using $\kappa_{X|k}$. This means that we are now able to define a *symmetric* morphism $\varphi : X \to X^t$ as one for which the dual morphism $\varphi^t : X = X^{tt} \to X^t$ equals φ. Note that the φ_L constructed above are symmetric by the Theorem. We can now finally define what a polarization is.

Definition 2.2. *Let $X \to \mathrm{Spec}\, k$ be an abelian variety with dual variety X^t and universal bundle (U, α_U). A* polarization *of X is a symmetric isogeny $\varphi : X \mapsto X^t$ such that $(1, \varphi)^* U|_{X \times X^t}$ is ample. If φ has degree 1 (as a map of varieties) then it is called a* principal polarization.

The following Proposition makes this somewhat more explicit.

Proposition 2.3. *Let $X \to \mathrm{Spec}\, k$ be an abelian variety.*

(i) *Any φ_L associated with an ample L is a polarization.*
(ii) *Conversely, let $\varphi : X \to X^t$ be a polarization. Then there is some finite extension K of k such that $\varphi \times 1_K : X \times_k K \to X \times_k K$ is the isogeny associated to an ample line bundle L on $X \times_k K$.*

The motivation to take 2.2 as Definition instead of, say, only allowing φ_L for L ample on X, is that it is possible that the isogeny φ_L can sometimes be descended further than the associated line bundle L.

We will now flesh this out for an elliptic curve over a field.

Proposition 2.4. *Let (E, O) be an elliptic curve over $S = \mathrm{Spec}\, k$. Then for any $d > 0$, E has a unique polarization of degree d^2.*

Proof. (Also see [Con04].)

Set $\overline{E} = E \times_k \overline{k}$, with zero section $\overline{O}$. First we have to construct the dual elliptic curve E^t, which strangely enough turns out to be E itself. This follows from a general result on autoduality of Jacobians, but in the special case under consideration, there is a simple proof using Riemann-Roch, which for elliptic curves states that over any extension field K of k, every degree 0 divisor in $E \times_k K$ is linearly equivalent to a unique divisor of the form $[O] - [P]$, with $P \in E(K)$. It proceeds as follows.

The zero section $O : \mathrm{Spec}\, k \to E$ gives rise to a morphism $O_E : \mathrm{Spec}\, k \times_k E \to E \times_k E$ after extending the base. By abuse of notation, we also denote the divisor on $E \times_k E$ that is the image of this section by O_E. Denote by Δ the divisor on $E \times_k E$ that is the image of the diagonal map. Now suggestively introduce

$$U = \mathcal{O}_{E \times E}([O_E] - [\Delta]).$$

This sheaf is rigidified along the zero section $E \times_k \mathrm{Spec}\, k \to E \times_k E$ (tedious check). We obtain a map $E \to E^t$ from the universal property of of E^t. On geometric points, this map is given by

$$x \mapsto (1_E \times x)^* U \cong \mathcal{O}_E([\overline{O}] - [x]).$$

(Note that we ignore the rigidification, as we are allowed to do by Proposition 1.2.) But as we remarked above, Riemann-Roch implies that this is a bijection from $E(\overline{k})$ to the line bundles of degree 0 on $\overline{E}$. Now note that since the degree does not vary in a connected family of line bundles, and the degree of the trivial bundle equals 0, this latter set of line bundles has to be the full set of points $E^t(\overline{k})$. Hence, by smoothness of the varieties involved, $E \to E^t$ is an isomorphism.

Now we determine the possible polarizations on E. As we have seen in the previous Proposition, over the algebraic closure $\overline{k}$ of k, any polarization φ comes from an ample line bundle L. So we have to determine φ_L for the $L = \mathcal{O}_{\overline{E}}(D)$ associated with some divisor D of positive degree d, which by Riemann-Roch is a finite sum $\sum_{i=1}^d y_i$ of $\overline{k}$-rational points y_i.

Let x and y be points of $\overline{E}$. First note that since t_x^* is pulling *back* under a translation, one has

$$t_x^*(\mathcal{O}_{\overline{E}}(D)) \cong \mathcal{O}_{\overline{E}}(t_{-x}(D)).$$

for any divisor D on $\overline{E}$. Also, by the group law on $\overline{E}$ we have that the divisors $[y] + [-y] + [\overline{O}]$ and $[y - x] + [x] + [-y]$ are both hyperplane sections of $\overline{E}$ after choosing a Weierstrass equation, hence equivalent to $3[\overline{O}]$. This implies that $[y - x] - [y]$ is equivalent to $[\overline{O}] - [x]$, or, using line bundles, that

$$\mathcal{O}_{\overline{E}}([y-x]) \otimes \mathcal{O}_{\overline{E}}([y]) \cong \mathcal{O}_{\overline{E}}([\overline{O}]) \otimes \mathcal{O}_{\overline{E}}([x])$$

Using similar techniques and induction, one shows that

$$(\mathcal{O}_{\overline{E}}([\overline{O}]) \otimes \mathcal{O}_{\overline{E}}([x])^{-1})^{\otimes d} \cong \mathcal{O}_{\overline{E}}([\overline{O}]) \otimes \mathcal{O}_{\overline{E}}([dx])^{-1}$$

Putting this together, we get

$$\begin{aligned}
\varphi_L(x) &= t_x^* \mathcal{O}_{\overline{E}}(\sum_{i=1}^d [y_i]) \otimes \mathcal{O}_{\overline{E}}(\sum_{i=1}^d [y_i])^{-1} \\
&\cong \bigotimes_{i=1}^d \mathcal{O}_{\overline{E}}([y_i - x]) \otimes \mathcal{O}_{\overline{E}}([y_i])^{-1} \\
&\cong (\mathcal{O}_{\overline{E}}([\overline{O}]) \otimes \mathcal{O}_{\overline{E}}([x])^{-1})^{\otimes d} \\
&\cong \mathcal{O}_{\overline{E}}([\overline{O}]) \otimes \mathcal{O}_{\overline{E}}([dx])^{-1} \\
&\cong \mathcal{O}_{\overline{E}}([\overline{O}] - [dx]).
\end{aligned}$$

So our polarization gives rise to the homomorphism $\overline{E} \to \overline{E}$ given on geometric points by $x \mapsto dx$, which is indeed of degree d^2 and clearly descends to E. We have proved the Proposition. □

Question: How does one describe the full scheme $\mathrm{Pic}^{\mathrm{rig}}_{X|k}$, along with the universal bundle on it? Hint: take a disjoint union of E^ts and modify (U, α_U) slightly on the new components.

A few more comments are perhaps in order.

(i) Note that $E^t(k)$ could be identified with the group of line bundles of degree 0 on E. This is no longer true in higher dimension: in general, the quotient of $\mathrm{Pic}(X)(k)$ by $X^t(k)$ is called the *Néron-Severi group*, and though generically it is isomorphic to $\mathbb{Z}$, it can be of arbitrarily high rank.

(ii) We have used that all degree divisors on E are of the form $[O] - [P]$, a special case of Theorem 1.5. Just to drive home the miraculousness of this result, note that among curves, this only happens for those of genus 1 with a rational point: it is certainly far from being true for arbitrary curves C of higher genus, since even if we suppose that there exists a rational point $0 \in C(k)$, then the map $P \mapsto [0] - [P]$ properly embeds C into its Jacobian, which is of higher dimension than C itself.

(iii) Finally, the Proposition showed that E always carries a principal polarization. This is false for more general abelian varieties, as we shall also see below.

More details for the results in the first two Sections are given in Chapters 2, 6, 7 and 11 of [EMvdG04].

3. THE CASE $k = \mathbb{C}$

Let X be an abelian variety over $\mathbb{C}$. Then the complex group manifold $X(\mathbb{C})^{\mathrm{an}}$ whose underlying set consists of the $\mathbb{C}$-points of X can be identified with a complex torus: that is to say,

$$X(\mathbb{C})^{\mathrm{an}} \cong V/\Lambda,$$

where V is a vector space over $\mathbb{C}$, Λ is a lattice in V (an abelian subgroup $\Lambda \hookrightarrow V$ with the property that the induced map $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to V$ is an isomorphism), and where the group structure on V/Λ is induced by addition on V. As is explained much more neatly and canonically in Chapter 1 of [Mum08], using this identification the group of line bundles on X can be described very explicitly.

A *sesquilinear form* on V is a map $S : V \times V \times \mathbb{C}$ that is $\mathbb{C}$-linear in the first factor and $\mathbb{C}$-antilinear in the second. This means that for $c \in C$ we have

$$s(cv, w) = cs(v, w) = s(v, \bar{c}w).$$

Such a form is called *positive definite* if $S(v, v) > 0$ for all v. A *Hermitian form* on V a sesquilinear form H that additionally satifies $H(v, w) = \overline{H(v, z)}$. A *pseudo-Riemann form* on V/Λ is an sesquilinear form on V whose imaginary part is $\mathbb{Z}$-valued on $\Lambda \times \Lambda$. A *Riemann form* is a pseudo-Riemann form that is Hermitian.

A word of warning: some authors insist on positive definiteness in the definition of a Riemann form, for reasons explained at the end of this section.

Given a Riemann form H, we can form the associated bilinear form and define an *H-character* as a function $\alpha : \Lambda \to \mathbb{C}^1 = \mathrm{Ker}(\mathbb{C}^\times \overset{\mathrm{Nm}}{\to} \mathbb{R}^\times)$ satisfying

$$\alpha(\lambda_1 + \lambda_2) = \exp\left(\pi i \mathrm{Im} H(\lambda_1, \lambda_2)\right)\alpha(\lambda_1)\alpha(\lambda_2).$$

Given H, there always exist H-characters for it.

Consider the set of pairs (H, α), where H is an Riemann form on V and α is an H-character. We can make this set into a group, call it G_{AH}, by taking $(0, 1)$ as the zero element and defining multiplication by

$$(H_1, \alpha_1)(H_2, \alpha_2) = (H_1 + H_2, \alpha_1\alpha_2).$$

By GAGA, the line bundles on X can be identified with those on $X(\mathbb{C})^{\mathrm{an}}$. We construct the latter geometrically (that is, as sections of a morphism). Take the trivial bundle $V \times \mathbb{C}$ on V. We have Λ act on the base V by $v \cdot \lambda \mapsto v + \lambda$, and on $V \times \mathbb{C}$ by

$$(v, z) \cdot \lambda = (v + \lambda, \exp(\pi H(v, \lambda) + \frac{1}{2}\pi H(\lambda, \lambda))\alpha(\lambda)z).$$

This gives a map

$$(V \times \mathbb{C})/\Lambda \longrightarrow V/\Lambda$$

that gives rise to a line bundle on V/Λ that we denote by $L(H, \alpha)$.

Theorem 3.1 (Appell-Humbert)**.** *The procedure above gives rise to an isomorphism*

$$G_{AH} \xrightarrow{\sim} \mathrm{Pic}(X).$$

The bundle $L(H, \alpha)$ is ample if and only if H is positive definite.

This allows us to describe the dual abelian variety. The points over $\mathbb{C}$ of this variety should parametrize the invertible sheaves on X that are algebraically equivalent to zero. Using Theorem 3.1, one shows that these are exactly the $L(0,\alpha)$, where now the α are honest characters $\Lambda \to \mathbb{C}^1$. Now any such character is of the form

$$\lambda \longmapsto \exp(2\pi i \mathrm{Im}(v^t(\lambda))),$$

where v^t is an element of the vector space $V^t = \mathrm{Hom}_{\overline{\mathbb{C}}}(V,\mathbb{C})$ of antilinear maps from V to C. Two v^t give rise to the same map if and only if their difference is in the abelian subgroup

$$\Lambda^t = \{v^t \, : \, \mathrm{Im}(v^t(\Lambda)) \subseteq \mathbb{Z}\}.$$

This subgroup of V^t is in fact a lattice. We have shown that for the complex torus V^t/Λ^t, there exists an isomorphism

$$\begin{aligned} V^t/\Lambda^t &\longrightarrow X^t(\mathbb{C}) \\ [v^t] &\longmapsto L(0, \exp(2\pi i \, \mathrm{Im}(v^t \cdot \_))). \end{aligned}$$

One can show that there exists a bundle on $V/\Lambda \times V^t/\Lambda^t$ whose fibers over points of V^t/Λ^t are exactly the $L(0,\alpha)$. Arguing as in Proposition 2.4, we see that V^t/Λ^t is the complex group manifold $X^t(\mathbb{C})^{\mathrm{an}}$ associated to the dual abelian variety X^t of X.

We can now also describe the polarizations of X. Proposition 2.3 tells us that these all come from ample line bundles on X. Theorem 3.1 states us that up to isomorphism, ample bundles are of the form $L(H,\alpha)$ with H positive definite. Again by Proposition 2.3, two bundles $L(H_1,\alpha_1)$ and $L(H_2,\alpha_2)$ give rise to the same polarization if and only if $L(H_1,\alpha_1) \otimes L(H_2,\alpha_2)^{-1} = L(0,\alpha)$ for some α, that is, if and only if $H_1 = H_2$. We get

Proposition 3.2. *Let X be an abelian variety over $\mathbb{C}$, and choose an isomorphism of complex group manifolds*

$$X(\mathbb{C})^{an} \cong V/\Lambda.$$

Then the polarizations of X correspond bijectively with the positive definite Hermitian forms H on V that take integral values on Λ.

Let $L = L(H,\alpha)$. The homomorphism $\varphi_L : X(\mathbb{C}) \mapsto X^t(\mathbb{C})$ is then given on the level of complex tori by

$$\begin{aligned} V/\Lambda &\longrightarrow V^t/\Lambda^t \\ [v] &\longmapsto [H(v, \_)]. \end{aligned}$$

The following correspondences relate our results with Definition 2.2 and Theorem 1.5:

map $X \to X^t$	$\longleftrightarrow$	sesquilinear form integral on $\Lambda \times \Lambda$
symmetric map	$\longleftrightarrow$	Hermitian form integral on $\Lambda \times \Lambda$
non-degenerate line bundle	$\longleftrightarrow$	non-degenerate form
ample line bundle	$\longleftrightarrow$	positive-definite form

We have seen that a polarization on $X(\mathbb{C})^{\mathrm{an}} = V/\Lambda$ gives rise to a positive definite Hermitian form on V that takes integral values on Λ. The existence of such a form is a necessary and sufficient condition for a complex torus V/Λ to be the complex group manifold associated to an abelian variety over X. For complex tori of dimension 1 this condition is always fulfilled. Analyzing this as in [Mum08], however, one sees that in a Baire-categorical sense, almost all complex tori of higher dimension are non-algebraic.

4. A FINITENESS RESULT

Let $X \to \operatorname{Spec} k$ be an abelian variety over an arbitrary field, and denote $\operatorname{End}^0(X) = \mathbb{Q} \otimes \operatorname{End}(X)$. Let φ be a polarization. Then we have a map

$$\begin{aligned} \operatorname{End}^0(X) &\longrightarrow \operatorname{End}^0(X) \\ \alpha &\longmapsto \alpha^\dagger = \varphi^{-1}\alpha^t\varphi \end{aligned}$$

that gives an involution called the *Rosati involution* (with respect to φ) on $\operatorname{End}^0(X)$. Choose a prime ℓ different from the characteristic of k, and denote by $\operatorname{Tr}(\alpha)$ the trace of α acting on the Tate module $T_\ell X$. As Peter mentioned, this a rational number is independent of ℓ.

One has the following important

Lemma 4.1. *The map*

$$\begin{aligned} \operatorname{End}^0(X) \times \operatorname{End}^0(X) &\longrightarrow \mathbb{Q} \\ (\alpha, \beta) &\longmapsto \operatorname{Tr}(\alpha^\dagger\beta) \end{aligned}$$

is a bilinear pairing with respect to addition on $\operatorname{End}^0(X)$. *This pairing is positive definite.*

Definition 4.2. *A* polarized abelian variety *is a pair* (X, φ), *with* $X \to \operatorname{Spec} k$ *an abelian variety and* $\varphi : X \to X^t$ *a polarization.*

A morphism *of polarized abelian varieties* $(X_1, \varphi_1) \to (X_2, \varphi_2)$ *is a homomorphism of abelian varieties* $f : X_1 \to X_2$ *that satisfies* $\varphi_1 = f^t\varphi_2 f$.

Proposition 4.3. *A polarized abelian variety has only finitely many automorphisms.*

Proof. (Sketch.)

First we analyze the group $\operatorname{End}(X)$. This is by definition equal to $\operatorname{Hom}(X, X)$. Let Y be any abelian variety over k, and ℓ be prime not dividing the characteristic of k. One has an injection

$$\operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}_{\mathbb{Z}_\ell}(T_\ell X, T_\ell Y)$$

because the points of ℓ-power torsion are dense in X.

Better yet, though not as easy, the induced map

$$\mathbb{Z}_\ell \otimes \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}_{\mathbb{Z}_\ell}(T_\ell X, T_\ell Y)$$

is also injective. Taking two different ℓ, one sees that $\operatorname{Hom}(X, Y)$ is a torsion-free abelian group of finite rank.

If $\alpha \in \operatorname{End}(X)$ respects φ, then $\alpha^\dagger\alpha = 1$ by the very definition of the Rosati involution. Now $\operatorname{Tr}(1) = 2g$ since $T_\ell X$ is of dimension $2g$ over Z_ℓ. So α is in both in the sublattice $\operatorname{End}(X)$ of $\mathbb{R} \otimes \operatorname{End}_0(X)$ and in the subset of $\mathbb{R} \otimes \operatorname{End}_0(X)$ consisting of those β for which the quadratic form $\operatorname{Tr}(\beta^\dagger\beta)$ equals $2g$. Since this quadratic form is positive definite, this subset is compact: therefore there are only finitely many possibilities for α. $\square$

Any automorphism of X that fixes the n-torsion subscheme for $n \geq 3$ is in fact the identity: a simple proof of this can be found in [Mil08, Proposition 14.4(b)].

Proposition 4.3 motivates the adage that the correct analogue of an elliptic curve is not an abelian variety but a polarized abelian variety; only for the latter can a decent theory of moduli be developed. Another reason is the fact that a curve C cannot be recovered from its Jacobian (as in the case of elliptic curves) but only from the Jacobian equipped with a certain natural principal polarization coming from C.

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